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CHEVALLEY COHOMOLOGY FOR KONTSEVICH'S GRAPHS

DIDIER ARNAL, ANGELA GAMMELLA, AND MOHSEN MASMOUDI

ABSTRACT. We introduce the Chevalley cohomology for the graded Lie algebra $T_{poly}(\mathbb{R}^d)$ of polyvector fields on \mathbb{R}^d . This cohomology occurs naturally in the problem of construction and classification of formalities on the space \mathbb{R}^d . Considering only graph formalities, *i.e.* formalities defined with the help of graphs like in the original construction of Kontsevich in [K1, K2], we define, as in [AM] for the Hochschild cohomology, the Chevalley cohomology directly on spaces of graphs. More precisely, observing first a noteworthy property for the Kontsevich's explicit formality on \mathbb{R}^d , we restrict ourselves to graph formalities with that property. With this restriction, we obtain some simple expressions of the Chevalley coboundary operator, especially, we can write this cohomology directly on the space of purely aerial, non-oriented graphs. We give finally examples and applications.

1. INTRODUCTION

In this article, we study formalities on the space \mathbb{R}^d . A formality is a formal non linear mapping \mathcal{F} between two formal graded manifolds $T_{poly}(\mathbb{R}^d)[1]$ and $D_{poly}(\mathbb{R}^d)[1]$, intertwining their natural vector fields Q and Q' .

Here, $T_{poly}(\mathbb{R}^d)[1]$ (resp. $D_{poly}(\mathbb{R}^d)[1]$) denotes the space of polyvector fields (resp. polydifferential operators) on \mathbb{R}^d graded by $|\alpha| = degree(\alpha) = k - 2$ if α is a k -vector field (resp. $|D| = m - 2$ if D is an m -differential operator) $-[1]$ stands for this choice of translation on degrees- and viewed as a formal manifold (see [K1, K2]). The monomial functions $\alpha_1.\alpha_2.\dots.\alpha_n$ on $T_{poly}(\mathbb{R}^d)$ are elements of the space $S^n(T_{poly}(\mathbb{R}^d)[1])$ of symmetric n -polyvector fields on $T_{poly}(\mathbb{R}^d)[1]$ (that means $\alpha_2.\alpha_1 = (-1)^{|\alpha_1||\alpha_2|}\alpha_1.\alpha_2$). The manifold $T_{poly}(\mathbb{R}^d)[1]$ is equipped with the formal bilinear vector field $Q = Q_2$, defined with the help of the Schouten bracket $[\cdot, \cdot]_S$:

$$Q_2(\alpha_1.\alpha_2) = (-1)^{(|\alpha_1|-1)|\alpha_2|}[\alpha_1, \alpha_2]_S.$$

Similarly, $D_{poly}(\mathbb{R}^d)[1]$ is equipped with the formal vector field

$$Q' = Q'_1 + Q'_2,$$

defined by

$$Q'_1(D_1) = -d_H D_1, \quad Q'_2(D_1.D_2) = (-1)^{(|D_1|-1)|D_2|}[D_1, D_2]_G.$$

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Here, d_H denotes the usual Hochschild coboundary operator: if D is a m -differential operator,

$$d_H D(f_1, \dots, f_{m+1}) = f_1 D(f_2, \dots, f_{m+1}) - D(f_1 f_2, \dots, f_{m+1}) + \dots + (-1)^m D(f_1, \dots, f_m) f_{m+1}$$

and $[\cdot, \cdot]_G$ is the Gerstenhaber bracket.

A formality \mathcal{F} is then given by a sequence of mappings \mathcal{F}_n :

$$\mathcal{F}_n : S^n(T_{poly}(\mathbb{R}^d)[1]) \rightarrow D_{poly}(\mathbb{R}^d)[1],$$

homogeneous with degree 0 and such that the ‘formality equation’:

$$\begin{aligned} d_H(\mathcal{F}_n)(\alpha_1 \dots \alpha_n) &= \frac{1}{2} \sum_{I \sqcup J = \{1, \dots, n\}, |I| \neq 0, |J| \neq 0} \varepsilon_\alpha(I, J) Q'_2(\mathcal{F}_{|I|}(\alpha_I) \cdot \mathcal{F}_{|J|}(\alpha_J)) \\ &\quad - \frac{1}{2} \sum_{k \neq \ell} \varepsilon_\alpha(k\ell, 1 \dots \widehat{k\ell} \dots n) \mathcal{F}_{n-1}(Q_2(\alpha_k \cdot \alpha_\ell) \cdot \alpha_1 \dots \widehat{\alpha_k \alpha_\ell} \dots \alpha_n) \end{aligned}$$

holds. Here, if $I = \{i_1 < \dots < i_\ell\}$, the notation α_I means $\alpha_{i_1} \dots \alpha_{i_\ell}$.

We shall impose moreover that \mathcal{F}_1 is the canonical mapping $\mathcal{F}_1^{(0)}$ from $T_{poly}(\mathbb{R}^d)$ to $D_{poly}(\mathbb{R}^d)$ defined by

$$\mathcal{F}_1^{(0)}(\xi_1 \wedge \dots \wedge \xi_n)(f_1, \dots, f_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^n \xi_{\sigma(i)}(f_i),$$

for any vector fields ξ_k and any functions f_i .

Let us now choose a coordinate system (x_t) on \mathbb{R}^d . In [K2], M. Kontsevich built explicitly a formality \mathcal{U} for \mathbb{R}^d , using families of graphs drawn on configuration spaces. A graph Γ has aerial and terrestrial vertices. The aerial vertices are labelled p_1, \dots, p_n and elements of the Poincaré half plane

$$\mathcal{H} = \{z \in \mathbb{C}, \Im(z) > 0\}.$$

The terrestrial vertices $q_1 < \dots < q_m$ are on the real line. The edges of Γ are arrows starting from an aerial vertex, ending to a terrestrial or an aerial vertex; there are no arrow of the form $p_i \vec{p}_i$ and no multiple arrow. If we fix a total ordering O on the edges of Γ , we get an oriented graph (Γ, O) . We say that O is compatible if, for all i , the arrows starting from p_i are before those starting from p_{i+1} and denote by $GO_{n,m}$ the set of oriented graphs (Γ, O) with n labelled aerial vertices, m labelled terrestrial vertices and O compatible.

Let us consider such an oriented graph (Γ, O) in $GO_{n,m}$. Let us also suppose there are k_i edges starting from the vertex p_i ($1 \leq i \leq n$). In [K2], Kontsevich defines a natural operator $B_{(\Gamma, O)}$ assigning a m -differential operator $B_{(\Gamma, O)}(\alpha_1 \otimes \dots \otimes \alpha_n)$ to a n -uple $(\alpha_1, \dots, \alpha_n)$ of polyvector fields α_i . This operator vanishes except if, for each i , α_i belongs to $T_{poly}^{k_i-1}(\mathbb{R}^d)$ (α_i is a k_i -polyvector field). Let us first consider all the

multi-indexes $(t_1, \dots, t_{|k|})$ with $|k| = \sum k_i$ and $1 \leq t_r \leq d$ for all $1 \leq r \leq |k|$. We denote by $\text{end}(a)$ the set of edges arriving on the vertex a and, if these edges are $e_{i_1}, \dots, e_{i_\ell}$, by $\partial_{\text{end}(a)}$ the operator

$$\partial_{\text{end}(a)} = \frac{\partial^\ell}{\partial x_{t_{i_1}} \dots \partial x_{t_{i_\ell}}}.$$

Then, we denote by $\text{star}(p_i)$ the ordered set $e_{j_1}^i < \dots < e_{j_{k_i}}^i$ of edges starting from p_i and, if α_i is a k_i -vector field, by $\alpha_i^{\text{star}(p_i)}$ the following component of α_i :

$$\alpha_i^{\text{star}(p_i)} = \alpha_i^{t_{j_1} \dots t_{j_{k_i}}}.$$

Finally, if, for each i , α_i is a k_i -vector field,

$$B_{(\Gamma, O)}(\alpha_1 \otimes \dots \otimes \alpha_n)(f_1, \dots, f_m) = \sum_{1 \leq t_1, \dots, t_{|k|} \leq d} \prod_{i=1}^n \partial_{\text{end}(p_i)} \alpha_i^{\text{star}(p_i)} \prod_{j=1}^m \partial_{\text{end}(q_j)} f_j.$$

$B_{(\Gamma, O)}$ will be called the graph operator associated with (Γ, O) .

The explicit formality \mathcal{U} of Kontsevich can now be written as a sum $\mathcal{U} = \sum_n \mathcal{U}_n$

with

$$\mathcal{U}_n = \sum_{m \geq 0} \sum_{(\Gamma, O) \in GO_{n, m}} w_{(\Gamma, O)} B_{(\Gamma, O)},$$

where the coefficient $w_{(\Gamma, O)}$, the weight of (Γ, O) is an integral on a compactified configuration space. To be precise, for $2n + m - 2 \geq 0$, let $\text{Conf}(n, m)$ be the space of n distinct points p_i in \mathcal{H} and m distinct points q_j on the real line $\partial\mathcal{H}$. Let us act on $\text{Conf}(n, m)$ by the group G of transformations $z \mapsto az + b$ ($a > 0, b$ real). Consider the quotient space

$$C_{n, m} = \text{Conf}(n, m)/G.$$

In [K2], Kontsevich associates with each oriented graph (Γ, O) the following form $\omega_{(\Gamma, O)}$ on $C_{n, m}$:

$$\omega_{(\Gamma, O)} = \frac{1}{k!} \bigwedge_{i=1}^n \left(d\Phi_{e_1^i} \wedge \dots \wedge d\Phi_{e_{k_i}^i} \right)$$

here $\{e_1^i < e_2^i < \dots < e_{k_i}^i\}$ denotes the ordered set $\text{star}(p_i)$ formed by the k_i edges starting from p_i , $k! := k_1! \dots k_n!$ and, if $e_\ell^i = p_i \vec{a}$,

$$\Phi_{e_\ell^i} = \Phi_{p_i \vec{a}} = \frac{1}{2\pi} \text{Arg} \left(\frac{a - p_i}{a - \bar{p}_i} \right).$$

The weight $w_{(\Gamma, O)}$ is then defined as the value of the integral $\omega_{(\Gamma, O)}$ on the connected component $C_{n, m}^+$ of $C_{n, m}$ for which $q_1 < \dots < q_m$.

In this work, we are looking for graph formalities, *i.e.* formalities on the space \mathbb{R}^d of the form $\mathcal{F} = \sum_n \mathcal{F}_n$ where the \mathcal{F}_n are homogeneous mapping (with degree 0) of the form

$$\mathcal{F}_n = \sum_{m \geq 0} \sum_{(\Gamma, O) \in GO_{n,m}} c_{(\Gamma, O)} B_{(\Gamma, O)},$$

with real coefficients $c_{(\Gamma, O)}$. We shall use the notation $\mathcal{F}_n = B_{\gamma_n}$ where γ_n is the linear combination

$$\gamma_n = \sum_{m \geq 0} \sum_{(\Gamma, O) \in GO_{n,m}} c_{(\Gamma, O)} (\Gamma, O).$$

Let us assume now we found $\mathcal{F}_1, \dots, \mathcal{F}_{n-1}$ (with $\mathcal{F}_1 = \mathcal{F}_1^{(0)} = \mathcal{U}_1$) such that the formality equation holds up to order $n-1$. Then, the next term \mathcal{F}_n , if it exists, is a solution of an equation:

$$d_H \circ \mathcal{F}_n = E_n,$$

that is

$$d_H(\mathcal{F}_n(\alpha_1 \dots \alpha_n)) = E_n(\alpha_1 \dots \alpha_n) = E_n(\alpha_{\{1, \dots, n\}}),$$

where $E_n(\alpha_{\{1, \dots, n\}})$ is a Hochschild cocycle. It is well-known that the Hochschild cohomology is localized in $T_{poly}(\mathbb{R}^d)[1]$. More precisely, the total skewsymmetrization $\mathfrak{a} \circ E_n(\alpha_{\{1, \dots, n\}})$ of $E_n(\alpha_{\{1, \dots, n\}})$ is a polydifferential operator of order $1, \dots, 1$, that is the image by $\mathcal{F}_1^{(0)}$ of a polyvector field. Moreover, there exists an operator A_n such that

$$E_n(\alpha_{\{1, \dots, n\}}) = (\mathfrak{a} \circ E_n + d_H \circ A_n)(\alpha_{\{1, \dots, n\}}).$$

Put now

$$\varphi_n = \mathcal{F}_1^{-1} \circ \mathfrak{a} \circ E_n,$$

that is

$$\varphi_n(\alpha_{\{1, \dots, n\}}) = \mathcal{F}_1^{-1} \left(\mathfrak{a} \left(E_n(\alpha_{\{1, \dots, n\}}) \right) \right),$$

then $\varphi_n : S^n(T_{poly}(\mathbb{R}^d)[1]) \rightarrow T_{poly}(\mathbb{R}^d)[1]$ is homogeneous with degree $|\varphi_n| = 1$.

In section 2, we define the Chevalley coboundary operator ∂ on $T_{poly}(\mathbb{R}^d)$. Then, we show that the mapping φ_n described above is a Chevalley cocycle, and, if it is a coboundary *i.e.* $\varphi_n = \partial \phi_{n-1}$, we can add to \mathcal{F}_{n-1} a Hochschild coboundary so that $\mathfrak{a}(E_n)$ vanishes and thus find a \mathcal{F}_n for which the formality equation holds up to order n .

In Section 3, we establish a remarkable property for the Kontsevich's weights. For any graph Γ (with k_i edges starting from p_i), denote by Δ the purely aerial graph obtained by cutting the legs $p_i \vec{q}_j$ and the feet q_j of Γ and by ℓ_i the number of aerial edges starting from p_i . We prove that:

$$\mathfrak{a} \left(\sum_{(\Gamma, O) \in GO_{n,m}^{(1)}} w_{(\Gamma, O)} B_{(\Gamma, O)} \right) = \sum_{(\Delta, O_\Delta) \in GO_n^{(0)}} w_{(\Delta, O_\Delta)} \frac{1}{m!} \sum_{GO_{n,m}^{(1)} \ni (\Gamma, O) \supset (\Delta, O_\Delta)} \frac{\ell!}{k!} \varepsilon(\Gamma) B_{(\Gamma, O)}.$$

Here $GO_{n,m}^{(1)}$ denotes the subspace of $GO_{n,m}$ formed by the oriented graphs having exactly one leg for each foot, $GO_n^{(0)}$ is the set of purely aerial oriented graphs (Δ, O_Δ) with n aerial vertices and O_Δ compatible and $\varepsilon(\Gamma)$ is an explicit sign depending only on Γ .

This property suggests us to study what we call K -graph formalities. A K -graph formality up to order n is a graph formality \mathcal{F} at order $n-1$ such that $\varphi_n = \mathcal{F}_1^{-1} \circ \mathbf{a} \circ E_n$ has the form $\varphi_n = \sum_{(\Delta, O_\Delta) \in GO_n^{(0)}} c_{(\Delta, O_\Delta)} C_{(\Delta, O_\Delta)}$ with real coefficients $c_{(\Delta, O_\Delta)}$ and where

$$C_{(\Delta, O_\Delta)} = \sum_{m \geq 0} \frac{1}{m!} \sum_{GO_{n,m}^{(1)} \ni (\Gamma, O) \supset (\Delta, O_\Delta)} \frac{\ell!}{k!} \varepsilon(\Gamma) B_{(\Gamma, O)}.$$

In Section 4, we first give some simple expressions of our Chevalley coboundary operator. Then, we restrict ourselves to K -graph formalities and study the Chevalley cohomology related to the question of building such formalities.

In Section 5, we show that the coboundary operator ∂ can be written directly on the aerial part of the graphs.

We devote the last section to explicit computations and applications. In particular, we prove the triviality of the cohomology for small value of n and give the restriction of the cohomology for linear formalities.

2. CHEVALLEY COHOMOLOGY AND FORMALITIES

In this section, we first define a graded Chevalley cohomology in a general algebraic setting i.e. for cochains $C : S^n(\mathfrak{g}[1]) \rightarrow \mathfrak{M}[1]$ where \mathfrak{g} is a graded Lie algebra and \mathfrak{M} a graded \mathfrak{g} -module. In fact two Chevalley coboundary operators are naturally associated with the formality equation for \mathbb{R}^d . The first one, say ∂' , is obtained by endowing $D_{poly}(\mathbb{R}^d)$ with a $T_{poly}(\mathbb{R}^d)$ -graded module structure, cochains are mappings $C : S^n(T_{poly}(\mathbb{R}^d)[1]) \rightarrow D_{poly}(\mathbb{R}^d)[1]$. The other one, ∂ , is obtained by considering $T_{poly}(\mathbb{R}^d)$ as a graded module over itself, cochains are mapping $C : S^n(T_{poly}(\mathbb{R}^d)[1]) \rightarrow T_{poly}(\mathbb{R}^d)[1]$. Using both ∂ and ∂' , we show that the obstructions to formalities can be interpreted as cocycles for ∂ .

2.1. Chevalley cohomology.

Let $(\mathfrak{g}, [\cdot, \cdot])$ be a graded Lie algebra and \mathfrak{M} a graded module over \mathfrak{g} . For reasons of homogeneity, we prefer to work with $\mathfrak{g}[1]$ and $\mathfrak{M}[1]$. Thus, we replace $[\cdot, \cdot]$ and the action of \mathfrak{g} on \mathfrak{M} respectively by $[\cdot, \cdot]'$ and $[\cdot, \cdot]_{\mathfrak{M}}$, defined for homogeneous α, β in $\mathfrak{g}[1]$

and m in $\mathfrak{M}[1]$ with degree $|\alpha|$ (resp. $|\beta|$, $|m|$) by

$$\begin{aligned} [\alpha, \beta]' &= (-1)^{(|\alpha|+1)|\beta|} [\alpha, \beta] \\ [\alpha, m]\mathfrak{M} &= (-1)^{(|\alpha|+1)|m|} \alpha.m. \end{aligned}$$

The space $C^n(\mathfrak{g}, \mathfrak{M})$ of n -cochains consists of mappings C from $S^n(\mathfrak{g}[1])$ to $\mathfrak{M}[1]$. The Chevalley coboundary ∂C of a n -cochain C , homogeneous with degree $|C|$, is the $n+1$ -cochain defined as

$$\begin{aligned} \partial C(\alpha_1 \dots \alpha_{n+1}) &= \\ &\sum_{i=1}^{n+1} (-1)^{|C||\alpha_i|} \varepsilon_\alpha(i, 1 \dots \hat{i} \dots n+1) [\alpha_i, C(\alpha_1 \dots \hat{\alpha}_i \dots \alpha_{n+1})] \mathfrak{M} \\ &- \frac{1}{2} \sum_{i \neq j} \varepsilon_\alpha(ij, 1 \dots \widehat{ij} \dots, n+1) (-1)^{|C|} C([\alpha_i, \alpha_j]' \cdot \alpha_1 \dots \widehat{\alpha_i \alpha_j} \dots \alpha_{n+1}). \end{aligned}$$

Here the α_i are homogeneous elements of \mathfrak{g} , $|\alpha_i|$ denotes the degree of α_i in $\mathfrak{g}[1]$ and for any permutation σ of $\{1, \dots, n\}$, $\varepsilon_\alpha(\sigma)$ is the sign of σ in the graded sense. We shall denote by $C_{[q]}^n(\mathfrak{g}, \mathfrak{M})$ the subspace of $C^n(\mathfrak{g}, \mathfrak{M})$ formed by the n -cochains of degree q and by $H_{[q]}^n(\mathfrak{g}, \mathfrak{M})$ the corresponding cohomology group. Note that ∂ sends $C_{[q]}^n(\mathfrak{g}, \mathfrak{M})$ into $C_{[q+1]}^{n+1}(\mathfrak{g}, \mathfrak{M})$.

Extending usual techniques to the graded case, it is possible to prove

Lemma 2.1. (See [Ga] for an explicit computation)

The operator ∂ is a cohomology operator i.e. $\partial^2 = \partial \circ \partial = 0$.

Let us now return to the graded Lie algebras $(T_{poly}(\mathbb{R}^d), [,]_S)$ and $(D_{poly}(\mathbb{R}^d), [,]_G)$ where $[,]_S$ is the Schouten bracket and $[,]_G$ the Gerstenhaber bracket. Let us precise first our conventions for these spaces and brackets.

Let α be a k -vector field and $\{e_i\}$ the canonical basis of \mathbb{R}^d . We put:

$$\begin{aligned} \alpha &= \sum_{j_1, \dots, j_k} \alpha^{j_1 \dots j_k} e_{j_1} \otimes \dots \otimes e_{j_k} = \sum_{j_1 < j_2 < \dots < j_k} \alpha^{j_1 \dots j_k} e_{j_1} \wedge \dots \wedge e_{j_k} \\ &= \frac{1}{k!} \sum_{j_1 \dots j_k} \alpha^{j_1 \dots j_k} e_{j_1} \wedge \dots \wedge e_{j_k}. \end{aligned}$$

For any k_1 -vector field α_1 and k_2 -vector field α_2 (the degree of α_i is $k_i - 1$ in $T_{poly}(\mathbb{R}^d)$), we define first a polyvector field $\alpha_1 \bullet \alpha_2$ with its components

$$\alpha_1 \bullet \alpha_2^{i_1 \dots i_{k_1+k_2-1}} = \frac{1}{k_1! k_2!} \sum_{\sigma \in S_{k_1+k_2-1}} \varepsilon(\sigma) \sum_{\ell=1}^{k_1} (-1)^{\ell-1} \sum_{s=1}^d \alpha_1^{i_{\sigma(1)} \dots i_{\sigma(\ell-1)} s i_{\sigma(\ell)} \dots i_{\sigma(k_1-1)}} \partial_s \alpha_2^{i_{\sigma(k_1)} \dots i_{\sigma(k_1+k_2-1)}}.$$

Now, $[\alpha_1, \alpha_2]_S$ can be written as

$$[\alpha_1, \alpha_2]_S = (-1)^{k_2(k_1-1)} \alpha_1 \bullet \alpha_2 - (-1)^{k_2-1} \alpha_2 \bullet \alpha_1.$$

Note that this choice for the Schouten bracket is denoted $[\cdot, \cdot]'_S$ in [AMM] and [MT].

On the other hand, for any m_1 -differential operator D_1 and any m_2 -differential operator D_2 (the degree of D_i is $m_i - 1$ in $D_{poly}(\mathbb{R}^d)$), we may write $[D_1, D_2]_G$ in the form

$$[D_1, D_2]_G = D_1 \circ D_2 - (-1)^{(m_1-1)(m_2-1)} D_2 \circ D_1$$

where

$$\begin{aligned} D_1 \circ D_2(f_1, \dots, f_{m_1+m_2-1}) = \\ \sum_{j=1}^{m_1} (-1)^{(m_2-1)(j-1)} D_1(f_1, \dots, f_{j-1}, D_2(f_j, \dots, f_{j+m_2-1}), f_{j+m_2}, \dots, f_{m_1+m_2-1}). \end{aligned}$$

Recall the canonical mapping $\mathcal{F}_1^{(0)}$ from $T_{poly}(\mathbb{R}^d)$ into $D_{poly}(\mathbb{R}^d)$: each k -vector field α can be viewed as a k -differential operator $\mathcal{F}_1^{(0)}(\alpha)$ of order $1, \dots, 1$:

$$(\mathcal{F}_1^{(0)}(\alpha))(f_1, \dots, f_k) = \langle \alpha, df_1 \wedge \dots \wedge df_k \rangle = \frac{1}{k!} \alpha^{i_1 \dots i_k} \partial_{i_1} f_1 \dots \partial_{i_k} f_k.$$

Now we consider the following action of $T_{poly}(\mathbb{R}^d)$:

$$\alpha.D = \mathfrak{a} \circ [\mathcal{F}_1^{(0)}(\alpha), D]_G \quad \forall \alpha \in T_{poly}(\mathbb{R}^d), \forall D \in D_{poly}(\mathbb{R}^d)$$

where \mathfrak{a} denotes the usual skewsymmetrization of differential operators and $[\cdot, \cdot]_G$ the Gerstenhaber bracket. This action defines a $T_{poly}(\mathbb{R}^d)$ -graded module structure on $D_{poly}(\mathbb{R}^d)$. Indeed, one can prove

Proposition 2.2.

The following equalities

$$\begin{aligned} i) \quad & \mathfrak{a} \circ [D_1, D_2]_G = \mathfrak{a} \circ [D_1, \mathfrak{a} \circ D_2]_G \\ ii) \quad & \mathcal{F}_1^{(0)}([\alpha_1, \alpha_2]_S) = \mathfrak{a} \circ [\mathcal{F}_1^{(0)}(\alpha_1), \mathcal{F}_1^{(0)}(\alpha_2)]_G \\ iii) \quad & \mathfrak{a} \circ [\mathcal{F}_1^{(0)}([\alpha_1, \alpha_2]_S), D]_G = \mathfrak{a} \circ [\mathcal{F}_1^{(0)}(\alpha_1), \mathfrak{a} \circ [\mathcal{F}_1^{(0)}(\alpha_2), D]_G]_G - \\ & \quad - (-1)^{(k_1-1)(k_2-1)} \mathfrak{a} \circ [\mathcal{F}_1^{(0)}(\alpha_2), \mathfrak{a} \circ [\mathcal{F}_1^{(0)}(\alpha_1), D]_G]_G \end{aligned}$$

hold for any D_1, D_2, D in $D_{poly}(\mathbb{R}^d)$, any k_1 -vector field α_1 and k_2 -vector field α_2 in $T_{poly}(\mathbb{R}^d)$. Especially, item (iii) means

$$[\alpha_1, \alpha_2]_S.D = \alpha_1.(\alpha_2.D) - (-1)^{(k_1-1)(k_2-1)} \alpha_2.(\alpha_1.D)$$

and thus $D_{poly}(\mathbb{R}^d)$ is a $T_{poly}(\mathbb{R}^d)$ -module.

Let us now endow $D_{poly}(\mathbb{R}^d)$ with the $T_{poly}(\mathbb{R}^d)$ -graded structure described above. If $C : \bigwedge^n(T_{poly}(\mathbb{R}^d)) = S^n(T_{poly}(\mathbb{R}^d)[1]) \rightarrow D_{poly}(\mathbb{R}^d)[1]$ is a mapping homogeneous with degree $|C|$, we can define its Chevalley coboundary $\partial' C$. The latter can be written using the vector fields Q and Q' , associated respectively with $T_{poly}(\mathbb{R}^d)$ and $D_{poly}(\mathbb{R}^d)$:

$$\begin{aligned} \partial' C(\alpha_1 \dots \alpha_{n+1}) = & \sum_{i=1}^{n+1} (-1)^{|C||\alpha_i|} \varepsilon_\alpha(i, 1 \dots \hat{i} \dots n+1) \mathfrak{a} \circ Q'_2 \left(\mathcal{F}_1^{(0)}(\alpha_i) \cdot C(\alpha_1 \dots \hat{\alpha}_i \dots \alpha_{n+1}) \right) \\ & - \frac{1}{2} \sum_{i \neq j} \varepsilon_\alpha(ij, 1 \dots \hat{i} \hat{j} \dots n+1) (-1)^{|C|} C \left(Q_2(\alpha_i \cdot \alpha_j) \cdot \alpha_1 \dots \widehat{\alpha_i \alpha_j} \dots \alpha_{n+1} \right). \end{aligned}$$

To simplify the writing, we will sometimes write α_i instead of $\mathcal{F}_1^{(0)}(\alpha_i)$.

On the other hand, considering $T_{poly}(\mathbb{R}^d)$ as a graded module over itself, one can define the Chevalley cohomology for $T_{poly}(\mathbb{R}^d)$. If $C : S^n(T_{poly}(\mathbb{R}^d)[1]) \rightarrow T_{poly}(\mathbb{R}^d)[1]$ is an n -cochain, homogeneous with degree $|C|$, its coboundary ∂C is:

$$\begin{aligned} \partial C(\alpha_1 \dots \alpha_{n+1}) = & \sum_{i=1}^{n+1} (-1)^{|C||\alpha_i|} \varepsilon_\alpha(i, 1 \dots \hat{i} \dots n+1) Q_2 \left(\alpha_i \cdot C(\alpha_1 \dots \hat{\alpha}_i \dots \alpha_{n+1}) \right) \\ & - \frac{1}{2} \sum_{i \neq j} \varepsilon_\alpha(ij, 1 \dots \hat{i} \hat{j} \dots n+1) (-1)^{|C|} C \left(Q_2(\alpha_i \cdot \alpha_j) \cdot \alpha_1 \dots \widehat{\alpha_i \alpha_j} \dots \alpha_{n+1} \right). \end{aligned}$$

Remark 2.1.

For any $\varphi : S^n(T_{poly}(\mathbb{R}^d)[1]) \rightarrow T_{poly}(\mathbb{R}^d)[1]$, we have:

$$\partial'(\mathcal{F}_1^{(0)} \circ \varphi) = \mathcal{F}_1^{(0)} \circ \partial \varphi.$$

2.2. Obstruction to formalities.

The two Chevalley coboundary operators ∂ and ∂' enable us to reformulate the formality equation. Indeed, suppose we want to construct a formality \mathcal{F} from $T_{poly}(\mathbb{R}^d)$ to $D_{poly}(\mathbb{R}^d)$. We need thus to solve recursively the formality equation (see [K, AMM] for notations)

$$\begin{aligned} d_H(\mathcal{F}_n)(\alpha_1 \dots \alpha_n) = & \frac{1}{2} \sum_{I \sqcup J = \{1, \dots, n\}, |I| \geq 1, |J| \geq 1} \varepsilon_\alpha(I, J) Q'_2 \left(\mathcal{F}_{|I|}(\alpha_I) \cdot \mathcal{F}_{|J|}(\alpha_J) \right) \\ & - \frac{1}{2} \sum_{k \neq \ell} \varepsilon_\alpha(k\ell, 1 \dots \hat{k} \hat{\ell} \dots n) \mathcal{F}_{n-1} \left(Q_2(\alpha_k \cdot \alpha_\ell) \cdot \alpha_1 \dots \widehat{\alpha_k \alpha_\ell} \dots \alpha_n \right), \end{aligned}$$

where d_H is the Hochschild coboundary operator.

Let us impose the first component \mathcal{F}_1 to be $\mathcal{F}_1^{(0)}$. Assume there exist mappings $\mathcal{F}_2, \dots, \mathcal{F}_{n-1}$, homogeneous with degree 0, and satisfying the formality equation up to order $n-1$. Denote by E_n the right hand side of the equation at the order n . Then E_n is a Hochschild cocycle: $d_H E_n = 0$ (see [AM] for instance). Thus

$$E_n = \mathfrak{a} \circ E_n + d_H C,$$

where $\mathfrak{a} \circ E_n$ is a differential operator of order 1, \dots , 1 and E_n is a coboundary if and only if $\mathfrak{a} \circ E_n = 0$. But:

$$\mathfrak{a} \circ E_n(\alpha_1 \dots \alpha_n) = \partial' \mathfrak{a} \mathcal{F}_{n-1}(\alpha_1 \dots \alpha_n) + \mathfrak{a} R_n(\alpha_1 \dots \alpha_n),$$

where

$$R_n(\alpha_1 \dots \alpha_n) = \frac{1}{2} \sum_{I \sqcup J = \{1, \dots, n\}, |I| \geq 2, |J| \geq 2} \varepsilon_\alpha(I, J) Q'_2(\mathcal{F}_{|I|}(\alpha_I) \cdot \mathcal{F}_{|J|}(\alpha_J)).$$

It follows directly from this expression that the degree of R_n and $\mathfrak{a} \circ R_n$ are both 1, $|R_n| = |\mathfrak{a} \circ R_n| = 1$. Moreover,

Theorem 2.3.

The skewsymmetrization $\mathfrak{a} \circ E_n$ of E_n can be identified through the inverse mapping of \mathcal{F}_1 with a ∂ -cocycle. If this cocycle is exact, we can find \mathcal{F}'_{n-1} and \mathcal{F}'_n , homogeneous with degree 0, such that $\mathcal{F}_2, \dots, \mathcal{F}_{n-2}, \mathcal{F}'_{n-1}, \mathcal{F}'_n$ satisfy the formality equation up to order n .

Proof: The proof proceed in three steps.

1. Let us first see that $\mathfrak{a} \circ R_n$ is a cocycle for ∂' :

$$\begin{aligned} \partial' \mathfrak{a} R_n(\alpha_1 \dots \alpha_{n+1}) &= \\ &= \sum_{i=1}^{n+1} (-1)^{|\alpha_i|} \varepsilon_\alpha(i, 1 \dots \hat{i} \dots n+1) \mathfrak{a} Q'_2(\alpha_i \cdot \mathfrak{a} R_n(\alpha_1 \dots \widehat{\alpha_i} \dots \alpha_{n+1})) \\ &\quad + \frac{1}{2} \sum_{i \neq j} \varepsilon_\alpha(ij, 1 \dots \widehat{ij} \dots n+1) \mathfrak{a} R_n(Q_2(\alpha_i \cdot \alpha_j) \cdot \alpha_1 \dots \widehat{\alpha_i \alpha_j} \dots \alpha_{n+1}) \\ &= \frac{1}{2} \sum_{i=1}^{n+1} (-1)^{|\alpha_i|} \varepsilon_\alpha(i, 1 \dots \hat{i} \dots n+1) \sum_{I \sqcup J = \{1 \dots \hat{i} \dots n+1\}, |I| \geq 2, |J| \geq 2} \varepsilon_{\alpha'}(I, J) \\ &\quad \mathfrak{a} Q'_2(\alpha_i \cdot \mathfrak{a} Q'_2(\mathcal{F}_{|I|}(\alpha_I) \cdot \mathcal{F}_{|J|}(\alpha_J))) \\ &\quad + \frac{1}{4} \sum_{i \neq j} \varepsilon_\alpha(ij, 1 \dots \widehat{ij} \dots n+1) \sum_{I \sqcup J = \{0, 1 \dots \widehat{ij} \dots n+1\}, |I| \geq 2, |J| \geq 2} \varepsilon_{\alpha''}(I, J) \\ &\quad \mathfrak{a} Q'_2(\mathcal{F}_{|I|}(\alpha_I) \cdot \mathcal{F}_{|J|}(\alpha_J)) \end{aligned}$$

$$= \frac{1}{2}(I) + \frac{1}{4}(II),$$

here $\alpha_0 := Q_2(\alpha_i, \alpha_j)$, $\varepsilon_{\alpha'} := \varepsilon_{\alpha \setminus \{\alpha_i\}}$ and $\varepsilon_{\alpha''} := \varepsilon_{(\alpha \cup \{\alpha_o\}) \setminus \{\alpha_i, \alpha_j\}}$.

The first term is:

$$(I) = \sum_{i=1}^{n+1} \sum_{\substack{I \sqcup J = \{1 \dots \hat{i} \dots n+1\} \\ |I| \geq 2, |J| \geq 2}} (-1)^{(|\alpha_I| + |\alpha_J|)|\alpha_i|} \varepsilon_{\alpha}(i, I, J) \mathfrak{a}Q'_2 \left(\mathfrak{a}Q'_2(\mathcal{F}_{|I|}(\alpha_I) \cdot \mathcal{F}_{|J|}(\alpha_J)) \cdot \alpha_i \right).$$

By Proposition 2.2, $\mathfrak{a}Q'_2$ satisfies the graded Jacobi identity, thus:

$$\begin{aligned} (I) &= - \sum_{i=1}^{n+1} \sum_{\substack{I \sqcup J = \{1 \dots \hat{i} \dots n+1\} \\ |I| \geq 2, |J| \geq 2}} (-1)^{|\alpha_J|(|\alpha_I| + |\alpha_i|)} \varepsilon_{\alpha}(i, I, J) \mathfrak{a}Q'_2 \left(\mathfrak{a}Q'_2(\mathcal{F}_{|J|}(\alpha_J) \cdot \alpha_i) \cdot \mathcal{F}_{|I|}(\alpha_I) \right) \\ &\quad - \sum_{\substack{I \sqcup J = \{1 \dots \hat{i} \dots n+1\} \\ |I| \geq 2, |J| \geq 2}} \varepsilon_{\alpha}(i, I, J) \mathfrak{a}Q'_2 \left(\mathfrak{a}Q'_2(\alpha_i \cdot \mathcal{F}_{|I|}(\alpha_I)) \cdot \mathcal{F}_{|J|}(\alpha_J) \right) \\ &= -2 \sum_{\substack{I \sqcup J = \{1 \dots \hat{i} \dots n+1\} \\ |I| \geq 2, |J| \geq 2}} \varepsilon_{\alpha}(i, I, J) \mathfrak{a}Q'_2 \left(\mathfrak{a}Q'_2(\alpha_i \cdot \mathcal{F}_{|I|}(\alpha_I)) \cdot \mathcal{F}_{|J|}(\alpha_J) \right). \end{aligned}$$

Similarly, the second term is:

$$\begin{aligned} (II) &= \sum_{i \neq j} \varepsilon_{\alpha}(ij, 1 \dots \hat{i} \hat{j} \dots n+1) \sum_{\substack{I \sqcup J = \{0, 1 \dots \hat{i} \hat{j} \dots n+1\} \\ |I| \geq 2, |J| \geq 2}} \varepsilon_{\alpha''}(I, J) \mathfrak{a}Q'_2 \left(\mathcal{F}_{|I|}(\alpha_I) \cdot \mathcal{F}_{|J|}(\alpha_J) \right) \\ &= \sum_{i \neq j} \sum_{\substack{I = I_1 \sqcup \{0\} \\ I_1 \sqcup J = \{1 \dots \hat{i} \hat{j} \dots n+1\}}} \varepsilon_{\alpha}(ij, I_1, J) \mathfrak{a}Q'_2 \left(\mathcal{F}_{|I|}(Q_2(\alpha_i, \alpha_j) \cdot \alpha_{I_1}) \cdot \mathcal{F}_{|J|}(\alpha_J) \right) \\ &\quad + \sum_{i \neq j} \sum_{\substack{J = J_1 \sqcup \{0\} \\ I \sqcup J_1 = \{1 \dots \hat{i} \hat{j} \dots n+1\}}} \varepsilon_{\alpha}(ij, 1 \dots \hat{i} \hat{j} \dots n+1) \varepsilon_{\alpha''}(I, \{0\}, J_1) \\ &\quad \mathfrak{a}Q'_2 \left(\mathcal{F}_{|I|}(\alpha_I) \cdot \mathcal{F}_{|J|}(Q_2(\alpha_i, \alpha_j) \cdot \alpha_{J_1}) \right) \\ &= \sum_{i \neq j} \left[\sum_{\substack{I = I_1 \sqcup \{0\} \\ I_1 \sqcup J = \{1 \dots \hat{i} \hat{j} \dots n+1\}}} \varepsilon_{\alpha}(ij, I_1, J) \mathfrak{a}Q'_2 \left(\mathcal{F}_{|I|}(Q_2(\alpha_i, \alpha_j) \cdot \alpha_{I_1}) \cdot \mathcal{F}_{|J|}(\alpha_J) \right) + \right. \\ &\quad \left. + \sum_{\substack{J = J_1 \sqcup \{0\} \\ I \sqcup J_1 = \{1 \dots \hat{i} \hat{j} \dots n+1\}}} \varepsilon_{\alpha}(ij, I, J_1) (-1)^{(|\alpha_i| + |\alpha_j| + 1)|\alpha_J|} \mathfrak{a}Q'_2 \left(\mathcal{F}_{|I|}(\alpha_I) \cdot \mathcal{F}_{|J|}(Q_2(\alpha_i, \alpha_j) \cdot \alpha_{J_1}) \right) \right] \end{aligned}$$

$$= 2 \sum_{i \neq j} \sum_{\substack{I=I_1 \sqcup \{0\} \\ I_1 \sqcup J = \{1 \dots \widehat{i} \dots n+1\}}} \varepsilon_\alpha(ij, I_1, J) \mathfrak{a}Q'_2 \left(\mathcal{F}_{|I|}(Q_2(\alpha_i \cdot \alpha_j) \cdot \alpha_{I_1}) \cdot \mathcal{F}_{|J|}(\alpha_J) \right).$$

Putting (I) and (II) together, we get:

$$\begin{aligned} \partial'(\mathfrak{a}R_n)(\alpha_1 \dots \alpha_{n+1}) &= \frac{1}{2}(I) + \frac{1}{4}(II) \\ &= \sum_{\substack{I' \sqcup J = \{1 \dots n+1\} \\ |J| \geq 2, |I'| \geq 3}} \varepsilon_\alpha(I', J) \left[- \sum_{i \in I' (I' = I_1 \sqcup \{i\})} \varepsilon_{\alpha_{\{i\} \sqcup I}}(i, I) \mathfrak{a}Q'_2 \left(\mathfrak{a}Q'_2(\alpha_i \cdot \mathcal{F}_{|I|}(\alpha_I)) \cdot \mathcal{F}_{|J|}(\alpha_J) \right) \right. \\ &\quad \left. + \frac{1}{2} \sum_{\substack{i \neq j \in I', I' = I_1 \sqcup \{ij\} \\ I = I_1 \sqcup \{0\}}} \varepsilon_{\alpha_{\{ij\} \sqcup I_1}}(ij, I_1) \mathfrak{a}Q'_2 \left(\mathcal{F}_{|I|}(Q_2(\alpha_i \cdot \alpha_j) \cdot \alpha_{I_1}) \cdot \mathcal{F}_{|J|}(\alpha_J) \right) \right]. \end{aligned}$$

Now, Proposition 2.2 and the definition of ∂' yield:

$$(*) \quad \partial'(\mathfrak{a}R_n)(\alpha_1 \dots \alpha_{n+1}) = - \sum_{\substack{I' \sqcup J = \{1 \dots n+1\} \\ |J| \geq 2, |I'| \geq 3}} \varepsilon_\alpha(I, J) \mathfrak{a}Q'_2 \left(\partial' \mathfrak{a} \mathcal{F}_{|I'-1|}(\alpha_{I'}) \cdot \mathcal{F}_{|J|}(\alpha_J) \right).$$

On the other hand, since the formality equation holds up to order $n-1$, we have:

$$\partial' \mathfrak{a} \mathcal{F}_{p-1} + \mathfrak{a}R_p = \mathfrak{a}(E_p) = \mathfrak{a}(d_H(\mathcal{F}_p)) = 0 \quad \forall p \leq n-1.$$

But $|I'| \leq n-1$ for all I' in the expression $(*)$, thus:

$$-\partial' \mathfrak{a} \mathcal{F}_{|I'-1|}(\alpha_{I'}) = \mathfrak{a}R_{|I'|}(\alpha_{I'}) = \frac{1}{2} \sum_{S \sqcup T = I', |S| \geq 2, |T| \geq 2} \varepsilon_{\alpha_{S \sqcup T}}(S, T) \mathfrak{a}Q'_2(\mathcal{F}_{|S|}(\alpha_S) \cdot \mathcal{F}_{|T|}(\alpha_T)).$$

Finally, $(*)$ becomes:

$$\begin{aligned} \partial'(\mathfrak{a}R_n)(\alpha_1 \dots \alpha_{n+1}) &= \\ &= \frac{1}{2} \sum_{\substack{S \sqcup T \sqcup J = \{1 \dots n+1\} \\ |S| \geq 2, |T| \geq 2, |J| \geq 2}} \varepsilon_\alpha(S \sqcup T, J) \varepsilon_{\alpha_{S \sqcup T}}(S, T) \mathfrak{a}Q'_2 \left(\mathfrak{a}Q'_2(\mathcal{F}_{|S|}(\alpha_S) \cdot \mathcal{F}_{|T|}(\alpha_T)) \cdot \mathcal{F}_{|J|}(\alpha_J) \right) \\ &= \frac{1}{2} \sum_{\substack{S \sqcup T \sqcup J = \{1 \dots n+1\} \\ |S| \geq 2, |T| \geq 2, |J| \geq 2}} \varepsilon_\alpha(S, T, J) \mathfrak{a}Q'_2 \left(\mathfrak{a}Q'_2(\mathcal{F}_{|S|}(\alpha_S) \cdot \mathcal{F}_{|T|}(\alpha_T)) \cdot \mathcal{F}_{|J|}(\alpha_J) \right) := S'. \end{aligned}$$

But thanks to the Jacobi identity, S' vanishes. Hence $\partial'(\mathfrak{a}R_n) = 0$ and $\partial'(\mathfrak{a}E_n) = 0$.

2. Let us put

$$\varphi_n = \mathcal{F}_1^{-1} \circ \mathfrak{a} \circ E_n.$$

Since

$$\partial'(\mathfrak{a} \circ E_n) = \partial' \mathcal{F}_1(\varphi_n) = \mathcal{F}_1(\partial \varphi_n) = 0,$$

φ_n is a cocycle for ∂ .

3. Let us assume that $\varphi_n = \partial\phi_{n-1}$ where $\phi_{n-1} : S^{n-1}(T_{poly}(\mathbb{R}^d)[1]) \rightarrow T_{poly}(\mathbb{R}^d)[1]$. Of course, $d_H\mathcal{F}_1(\phi_{n-1}) = 0$. Therefore, the mappings $\mathcal{F}'_2 = \mathcal{F}_2, \dots, \mathcal{F}'_{n-2} = \mathcal{F}_{n-2}$, $\mathcal{F}'_{n-1} = \mathcal{F}_{n-1} - \mathcal{F}_1 \circ \phi_{n-1}$ satisfy the formality equation up to order $n-1$. Moreover, the Hochschild cocycle E'_n corresponding to these \mathcal{F}'_p satisfies

$$\mathfrak{a} \circ E'_n = \mathfrak{a} \circ E_n - \partial'(\mathcal{F}_1 \circ \phi_{n-1}) = \mathfrak{a} \circ E_n - \mathcal{F}_1(\partial\phi_{n-1}) = 0.$$

We are now able to find \mathcal{F}'_n such that $E'_n = d_H\mathcal{F}'_n$. This ends the proof.

3. SKEWSYMMETRIZATION

The aim of this section is the proof of a noteworthy property of the Kontsevich's weights and the definition of K -graph formalities.

3.1. Skewsymmetrization and 1-graphs.

Let us first consider a m -differential operator D on \mathbb{R}^d , vanishing on constants. We can decompose D as

$$D = D^{(1)} + D^{(>1)},$$

where $D^{(1)}$ has order 1 in each of its arguments and $D^{(>1)}$ has order larger than 1 for at least one of its arguments. The skewsymmetrization $\mathfrak{a}(D)$ of D i.e.

$$\mathfrak{a}(D)(f_1, \dots, f_m) = \frac{1}{m!} \sum_{\sigma \in S_m} \varepsilon(\sigma) D(f_{\sigma^{-1}(1)}, \dots, f_{\sigma^{-1}(m)})$$

satisfies

$$\mathfrak{a}(D) = \mathfrak{a}(D^{(1)}) + \mathfrak{a}(D^{(>1)})$$

and therefore

$$\mathfrak{a}(D)^{(1)} = \mathfrak{a}(D^{(1)}).$$

We assume D to be defined with the help of graphs:

$$D_{(\alpha_1, \dots, \alpha_n)} = \sum_{\Gamma} c_{\Gamma} B_{\Gamma}(\alpha_1 \otimes \dots \otimes \alpha_n)$$

where the c_{Γ} are real. To compute $\mathfrak{a}(D)^{(1)}$, we need only to consider

$$D_{(\alpha_1, \dots, \alpha_n)}^{(1)} = \sum_{\Gamma \in G^{(1)}} c_{\Gamma} B_{\Gamma}(\alpha_1 \otimes \dots \otimes \alpha_n),$$

where $G^{(1)}$ denotes the family of 1-graphs i.e. graphs having exactly one leg for each foot.

However, as in [K2], to define B_{Γ} , we need to choose a total ordering O on the set $E(\Gamma)$ of edges of Γ . To be exact, we first choose a labelling on the aerial vertices of Γ , say p_1, \dots, p_n . Then we put away the arrows starting from p_1 , from p_2 ... and finally from p_n . We get a total ordering of $E(\Gamma)$ which is compatible with the ordering

$p_1 < p_2 < \dots < p_n$ in the sense that the arrows starting from p_i are before those starting from p_{i+1} .

From now on, we denote by $GO_{n,m}$ the set of oriented graphs (Γ, O) with n labelled aerial vertices, m labelled terrestrial vertices and compatible ordering O , and by $GO_{n,m}^{(1)}$ the subset of $GO_{n,m}$ formed by the oriented 1-graphs. In fact, our previous notations $\sum c_\Gamma B_\Gamma$ actually mean

$$\sum_{\Gamma} c_\Gamma B_\Gamma = \sum_{(\Gamma, O) \in GO_{n,m}} c_{(\Gamma, O)} B_{(\Gamma, O)},$$

and

$$\sum_{\Gamma \in G^{(1)}} c_\Gamma B_\Gamma = \sum_{(\Gamma, O) \in GO_{n,m}^{(1)}} c_{(\Gamma, O)} B_{(\Gamma, O)}.$$

3.2. A noteworthy property of Kontsevich's weights.

3.2.1. Kontsevich's weights.

Let (Γ, O) be an oriented graph in $GO_{n,m}^{(1)}$ with aerial vertices $p_1 < \dots < p_n$. We denote by k_i the number of edges starting from p_i . We moreover denote U_i (resp. V_i) the ordered set of legs (resp. aerial edges) starting from p_i . Let ℓ_i be the number of elements in V_i , U_i has $m_i = k_i - \ell_i$ elements. By definition of $GO_{n,m}^{(1)}$, the number of legs is exactly the number of terrestrial vertices i.e. $m = \sum_{i=1}^n m_i$.

Starting from (Γ, O) , it will be helpful to consider the permutation s_O defined by

$$s_O : E(\Gamma) \mapsto V_1 \cup \dots \cup V_n \cup U_1 \cup \dots \cup U_n.$$

After this permutation s_O , we get a new ordering O' on $E(\Gamma)$ (O' is no more compatible) such that all the legs are put at the end, and we can define a permutation τ_O of the legs of (Γ, O') by putting first the leg ending at q_1 , then the leg ending at $q_2 \dots$ and lastly the leg ending at q_m . Let us extend the permutation τ_O to $V_1 \cup \dots \cup V_n \cup U_1 \cup \dots \cup U_n$ by setting $\tau_O(v) = v$ for all v in $\bigcup V_i$. Finally, note Δ the aerial graph obtained from Γ by cutting the legs and the feet and by O_Δ the (compatible) ordering on Δ induced by O .

Let $GO_n^{(0)}$ be the set of oriented, purely aerial graphs (Δ, O_Δ) with n vertices.

With these notations, the Kontsevich's weight associated with (Γ, O) can be written as

$$w_{(\Gamma, O)} = \frac{1}{k!} \varepsilon(s_O) \varepsilon(\tau_O) \int_{C_{n,0}^+} \bigwedge_{r=1}^{|\ell|} d\Phi_{e_r^\Delta} \int_{\substack{q_1 < \dots < q_m \\ \text{oriented by} \\ dq_1 \wedge \dots \wedge dq_m}} \bigwedge_{j=1}^m d\Phi_{p_{i_j} \vec{q}_j},$$

where $k! = k_1! \dots k_n!$, $|\ell| := \sum \ell_i$, $V_1 \cup \dots \cup V_n := \{e_1^\Delta < \dots < e_{|\ell|}^\Delta\}$ and i_j stands for the unique index i such that the leg arriving on q_j is exactly $p_i \vec{q}_j$.

Note that the Kontsevich's weight of (Δ, O_Δ) is just

$$w_{(\Delta, O_\Delta)} = \frac{1}{\ell!} \int_{C_{n,0}^+} \bigwedge_{r=1}^{|\ell|} d\Phi_{e_r^\Delta},$$

($\ell! = \ell_1! \dots \ell_n!$). Thus, we have

$$w_{(\Gamma, O)} = w_{(\Delta, O_\Delta)} \frac{\ell!}{k!} \varepsilon(s_O) \varepsilon(\tau_O) \int_{\substack{q_1 < \dots < q_m \\ \text{oriented by} \\ dq_1 \wedge \dots \wedge dq_m}} \bigwedge_{j=1}^m d\Phi_{p_{i_j} \vec{q}_j}.$$

3.2.2. The S_m action on $GO_{n,m}^{(1)}$.

Let σ be an element in the permutation group S_m . With any graph (Γ, O) in $GO_{n,m}^{(1)}$, we associate a new graph $(\sigma(\Gamma), \sigma(O))$. We keep for $\sigma(\Gamma)$ the vertices of Γ . But, if $E(\Gamma) = \{e_1 < \dots < e_{|k|}\}$, we put $E(\sigma(\Gamma)) = \{e'_1 < \dots < e'_{|k|}\}$ where if e_r is an aerial edge, $e'_r := e_r$, and if $e_r = p_i \vec{q}_j$ is a leg, $e'_r := p_i \vec{q}_{\sigma(j)}$ (see Figure 1 below). In this way, we get a free action of S_m on $GO_{n,m}^{(1)}$.

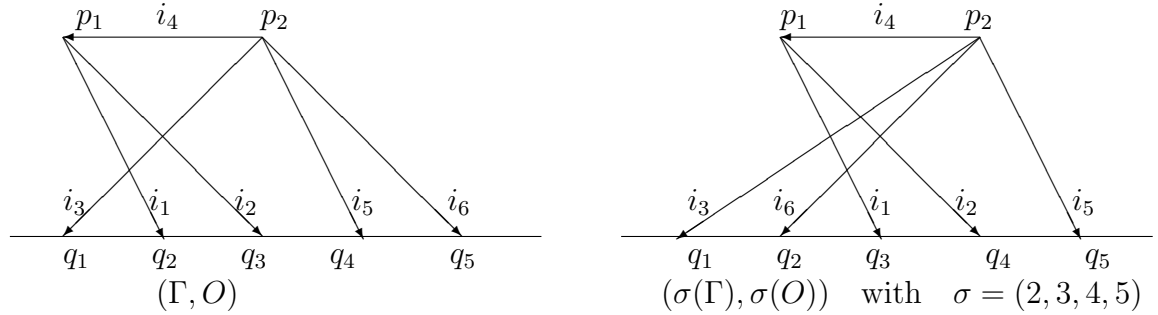


Figure 1

Lemma 3.1. *For all σ in S_m , for all (Γ, O) in $GO_{n,m}^{(1)}$,*

$$B_{(\sigma(\Gamma), \sigma(O))}(\alpha)(f_1, \dots, f_m) = B_{(\Gamma, O)}(\alpha)(f_{\sigma(1)}, \dots, f_{\sigma(m)}) \quad f_i \in C^\infty(\mathbb{R}^d).$$

Proof: Let us denote by r_j the label of the leg arriving on q_j in (Γ, O) . Then, in $(\sigma(\Gamma), \sigma(O))$, this leg has the same label r_j , but it ends at $q_{\sigma(j)}$. The aerial edges are kept unchanged. The result follows easily.

Lemma 3.2. *Let σ be in S_m and (Γ, O) in $GO_{n,m}^{(1)}$. Then,*

$$\varepsilon(s_{\sigma(O)}) = \varepsilon(s_O) \quad \text{and} \quad \varepsilon(\tau_{\sigma(O)}) = \varepsilon(\sigma)\varepsilon(\tau_O).$$

Proof: When building $(\sigma(\Gamma), \sigma(O))$, we get a bijective mapping from $E(\Gamma)$ into $E(\sigma(\Gamma))$, say $\tilde{\sigma}$. In fact, $s_{\sigma(O)} = \tilde{\sigma} \circ s_O \circ \tilde{\sigma}^{-1}$. Thus, $\varepsilon(s_{\sigma(O)}) = \varepsilon(s_O)$.

Let us now denote by $q_{a_1^i}, \dots, q_{a_{m_i}^i}$ the feet of the legs starting from p_i . By definition, τ_O is the permutation

$$p_1 \vec{q}_{a_1^1}, p_1 \vec{q}_{a_2^1}, \dots, p_n \vec{q}_{a_{m_n}^n} \mapsto p_{i_1} \vec{q}_1, \dots, p_{i_m} \vec{q}_m.$$

We may write:

$$\tau_O^{-1} : (1, \dots, m) \mapsto (a_1^1, \dots, a_{m_n}^n).$$

By definition of $(\sigma(\Gamma), \sigma(O))$, we have

$$\tau_{\sigma(O)}^{-1} : (1, \dots, m) \mapsto (\sigma(a_1^1), \dots, \sigma(a_{m_n}^n)).$$

Thus, $\tau_{\sigma(O)}^{-1} \circ \tau_O = \sigma$. The result follows.

3.2.3. A noteworthy property.

Proposition 3.3. *We keep our notations. Especially, for any (Γ, O) in $GO_{n,m}^{(1)}$ and any (Δ, O_Δ) in $GO_n^{(0)}$, we denote by $w_{(\Gamma, O)}$ and $w_{(\Delta, O_\Delta)}$ the corresponding weights. Then, we have*

$$\begin{aligned} \mathfrak{a}\left(\sum_{(\Gamma, O) \in GO_{n,m}^{(1)}} w_{(\Gamma, O)} B_{(\Gamma, O)}(\alpha)\right) &= \\ &= \sum_{(\Delta, O_\Delta) \in GO_n^{(0)}} w_{(\Delta, O_\Delta)} \sum_{m \geq 0} \frac{1}{m!} \sum_{(\Gamma, O) \supset (\Delta, O_\Delta), (\Gamma, O) \in GO_{n,m}^{(1)}} \frac{\ell!}{k!} \varepsilon(s_O) \varepsilon(\tau_O) B_{(\Gamma, O)}(\alpha). \end{aligned}$$

Proof: Skewsymmetrizing and using Lemma 3.1, we get

$$\begin{aligned} \mathfrak{a}\left(\sum_{(\Gamma, O) \in GO_{n,m}^{(1)}} w_{(\Gamma, O)} B_{(\Gamma, O)}(\alpha)\right)(f_1 \otimes \dots \otimes f_m) &= \\ &= \frac{1}{m!} \sum_{\sigma \in S_m} \varepsilon(\sigma) \sum_{(\Gamma, O) \in GO_{n,m}^{(1)}} w_{(\Gamma, O)} B_{(\Gamma, O)}(\alpha) (f_{\sigma^{-1}(1)} \otimes \dots \otimes f_{\sigma^{-1}(m)}) \\ &= \frac{1}{m!} \sum_{\sigma \in S_m} \varepsilon(\sigma) \sum_{(\Gamma, O) \in GO_{n,m}^{(1)}} w_{(\Gamma, O)} B_{(\sigma^{-1}(\Gamma), \sigma^{-1}(O))}(\alpha) (f_1 \otimes \dots \otimes f_m) \\ &= \frac{1}{m!} \sum_{\sigma \in S_m} \varepsilon(\sigma) \sum_{(\Gamma, O) \in GO_{n,m}^{(1)}} w_{(\sigma(\Gamma), \sigma(O))} B_{(\Gamma, O)}(\alpha) (f_1 \otimes \dots \otimes f_m). \end{aligned}$$

Now, by definition,

$$w_{(\sigma(\Gamma), \sigma(O))} = \varepsilon(s_{\sigma(O)}) \varepsilon(\tau_{\sigma(O)}) \frac{\ell!}{k!} \int_{C_{n,0}^+} \bigwedge_{r=1}^{|\ell|} d\Phi_{e_r^\Delta} \int_{\substack{q_1 < \dots < q_m \\ \text{oriented by} \\ dq_1 \wedge \dots \wedge dq_m}} \bigwedge_{j=1}^m d\Phi_{p_{i'_j} \vec{q}_j},$$

where i'_j stands for the unique index i' such that the leg arriving on q_j is exactly $p_{i'} \vec{q}_j$. Now

$$\bigwedge_{j=1}^m d\Phi_{p_{i'_j} \vec{q}_j} = \varepsilon(\sigma) \bigwedge_{j=1}^m d\Phi_{p_{i_j} \vec{q}_{\sigma(j)}},$$

then by Lemma 3.2:

$$w_{(\sigma(\Gamma), \sigma(O))} = \frac{\ell!}{k!} \varepsilon(s_O) \varepsilon(\tau_O) \int_{C_{n,0}^+} \bigwedge_{r=1}^{|\ell|} d\Phi_{e_r^\Delta} \int_{\substack{q_1 < \dots < q_m \\ \text{oriented by} \\ dq_1 \wedge \dots \wedge dq_m}} \bigwedge_{j=1}^m d\Phi_{p_{i_j} \vec{q}_{\sigma(j)}}.$$

With the new variables ($q'_j = q_{\sigma(j)}$), we get

$$w_{(\sigma(\Gamma), \sigma(O))} = \frac{\ell!}{k!} w_{(\Delta, O_\Delta)} \varepsilon(s_O) \varepsilon(\tau_O) \int_{D^\sigma} \bigwedge_{j=1}^m d\Phi_{p_{i_j} \vec{q}'_j}$$

where D^σ is the domain $q'_{\sigma^{-1}(1)} < \dots < q'_{\sigma^{-1}(m)}$ oriented by $dq'_1 \wedge \dots \wedge dq'_m$. Thus,

$$\begin{aligned} \mathfrak{a} \left(\sum_{(\Gamma, O) \in GO_{n,m}^{(1)}} w_{(\Gamma, O)} B_{(\Gamma, O)}(\alpha) \right) &= \\ &= \frac{1}{m!} \sum_{\sigma \in S_m} \sum_{(\Gamma, O) \in GO_{n,m}^{(1)}} w_{(\Delta, O_\Delta)} \frac{\ell!}{k!} \varepsilon(s_O) \varepsilon(\tau_O) \int_{D^\sigma} \bigwedge_{j=1}^m d\Phi_{p_{i_j} \vec{q}'_j} B_{(\Gamma, O)}(\alpha) \\ &= \sum_{(\Delta, O_\Delta) \in GO_n^{(0)}} w_{(\Delta, O_\Delta)} \frac{1}{m!} \sum_{\substack{(\Gamma, O) \in GO_{n,m}^{(1)} \\ (\Gamma, O) \supset (\Delta, O_\Delta)}} \frac{\ell!}{k!} \varepsilon(s_O) \varepsilon(\tau_O) \left(\sum_{\sigma \in S_m} \int_{D^\sigma} \bigwedge_{j=1}^m d\Phi_{p_{i_j} \vec{q}'_j} \right) B_{(\Gamma, O)}(\alpha) \\ &= \sum_{(\Delta, O_\Delta) \in GO_n^{(0)}} w_{(\Delta, O_\Delta)} \frac{1}{m!} \sum_{GO_{n,m}^{(1)} \ni (\Gamma, O) \supset (\Delta, O_\Delta)} \frac{\ell!}{k!} \varepsilon(s_O) \varepsilon(\tau_O) B_{(\Gamma, O)}(\alpha). \end{aligned}$$

This ends the proof.

3.3. K -graph formalities.

Let us consider the Kontsevich's explicit formality $\mathcal{U} = \sum_n \mathcal{U}_n$ on \mathbb{R}^d . If (Γ, O) is an oriented graph with O not compatible, we shall put as in [AMM]:

$$B_{(\Gamma, O)} = \varepsilon(\sigma_{(O, O_o)}) B_{(\Gamma, O_o)},$$

where O_o is any compatible orientation on Γ and $\sigma_{(O,O_o)}$ stands for the permutation of $E(\Gamma)$ obtained by changing (Γ, O) into (Γ, O_o) . We shall also put

$$\omega'_{(\Gamma,O)} = \frac{k!}{|k|!} \omega_{(\Gamma,O)} \quad \text{and} \quad w'_{(\Gamma,O)} = \int_{C_{n,m}^+} \omega'_{(\Gamma,O)},$$

where $k! = k_1! \dots k_n!$ and $|k| = \sum k_i$ if k_i denotes the number of edges starting from the vertex p_i of Γ , and $\omega_{(\Gamma,O)} = d\Phi_{e_1} \wedge \dots \wedge d\Phi_{e_{|k|}}$ if $E(\Gamma) = \{e_1 < \dots < e_{|k|}\}$.

From now on, we shall denote by $GO'_{n,m}$ the set of oriented graphs (Γ', O') , with O' not necessarily compatible. Then, we have

$$\mathcal{U}_n = \sum_{m \geq 0} \sum_{(\Gamma', O') \in GO'_{n,m}} w'_{(\Gamma', O')} B_{(\Gamma', O')}.$$

Let us write the formality equation for \mathcal{U} as:

$$F_n = E_n - d_H(\mathcal{U}_n) = 0.$$

Rewriting the proof of the formality theorem by Kontsevich, one can see that F_n looks like a sum over the faces F of the boundary $\partial C_{n,m}^+$ of $C_{n,m}^+$ (see [AMM] for details):

$$F_n = \sum_{m \geq 0} \sum_{F \subset \partial C_{n,m}^+} \sum_{(\Gamma', O') \in GO'_{n,m}} w'^F_{(\Gamma', O')} B_{(\Gamma', O')}$$

where $w'^F_{(\Gamma', O')}$ is the integral over F of the closed 2-form $\omega'_{(\Gamma', O')}$.

The fact that $F_n = 0$ follows then directly from the Stokes formula. Especially, we have $\mathfrak{a}(E_n) = 0$.

Now, we saw that $\mathfrak{a}(E_n) = \mathfrak{a}(E_n^{(1)})$. Thus, for a fixed face F of $\partial C_{n,m}^+$, the corresponding term in $\mathfrak{a}(E_n)$ is a sum over 1-graphs of the form:

$$\mathfrak{a}\left(\sum_{(\Gamma', O') \in GO'^{(1)}_{n,m}} w'^F_{(\Gamma', O')} B_{(\Gamma', O')}\right).$$

Each term of this sum satisfies our relation:

$$\begin{aligned} \mathfrak{a}\left(\sum_{(\Gamma', O') \in GO'^{(1)}_{n,m}} w'^F_{(\Gamma', O')} B_{(\Gamma', O')}(\alpha)\right) &= \\ &= \sum_{(\Delta, O_\Delta)} w^F_{(\Delta, O_\Delta)} \frac{1}{m!} \sum_{GO'^{(1)}_{n,m} \ni (\Gamma, O) \supset (\Delta, O_\Delta)} \frac{\ell!}{k!} \varepsilon(s_O) \varepsilon(\tau_O) B_{(\Gamma, O)}(\alpha) \end{aligned}$$

where $w^F_{(\Delta, O_\Delta)} = \int_F \omega_{(\Delta, O_\Delta)}$. Let us prove this precisely.

A face has either type 1 or type 2 (see [K2] or [AMM]). We consider only the faces such that $w'^F_{(\Gamma', O')}$ can be different from 0.

(i) If the face F has type 1: then two vertices p_i, p_j of Γ' , related by exactly one edge, are collapsing and the face is $F = C_{\{p_i, p_j\}} \times C^+_{\{p, p_1, \dots, \widehat{p_i p_j}, \dots, p_n\}; \{q_1, \dots, q_m\}}$. We

parametrize $C_{n,m}^+$ by

$$\rho = \frac{|p_j - p_i|}{\Im(p_i)} \quad p'_j = \frac{p_j - p_i}{|p_j - p_i|} \quad p'_r = \frac{p_r - \Re(p_i)}{\Im(p_i)} \quad q'_s = \frac{q_s - \Re(p_i)}{\Im(p_i)}.$$

With the signs computed in [AMM], we can write

$$w'_{(\Gamma', O')}^F = - \int_{C_2} d\Phi_{p_i \vec{p}_j} \int_{C_{n-1,m}^+} \omega'_{(\Gamma_2, O_2)}$$

where Γ_2 is the graph obtained from Γ' by gluing together p_i and p_j on the point p , and suppressing the edge $p_i \vec{p}_j$. This weight $w'_{(\Gamma', O')}^F$ corresponds actually to a limit when ρ tends to zero. In fact, if we put

$$C_{n,m}^+(\varepsilon) = C_{n,m}^+ \cap \{(p, q) : \rho = \varepsilon\},$$

we get

$$w'_{(\Gamma', O')}^F = \lim_{\varepsilon \rightarrow 0} \frac{k!}{|k|!} \int_{C_{n,m}^+(\varepsilon)} \omega_{(\Gamma', O')} := \lim_{\varepsilon \rightarrow 0} w'_{(\Gamma', O')}^F(\varepsilon).$$

This limit vanishes for graphs (Γ', O') whose vertices p_i and p_j are linked by two edges or no edges at all. We can thus also consider these graphs in our sum. Then,

$$\mathfrak{a}\left(\sum_{(\Gamma', O') \in GO_{n,m}^{(1)}} w'_{(\Gamma', O')}^F B_{(\Gamma', O')}(\alpha)\right) = \lim_{\varepsilon \rightarrow 0} \mathfrak{a}\left(\sum_{(\Gamma', O') \in GO_{n,m}^{(1)}} w'_{(\Gamma', O')}^F(\varepsilon) B_{(\Gamma', O')}(\alpha)\right).$$

Passing to compatible orderings, we obtain:

$$\mathfrak{a}\left(\sum_{(\Gamma', O') \in GO_{n,m}^{(1)}} w'_{(\Gamma', O')}^F B_{(\Gamma', O')}(\alpha)\right) = \lim_{\varepsilon \rightarrow 0} \mathfrak{a}\left(\sum_{(\Gamma, O) \in GO_{n,m}^{(1)}} w_{(\Gamma, O)}^F(\varepsilon) B_{(\Gamma, O)}(\alpha)\right).$$

By Proposition 3.3, we get as announced:

$$\begin{aligned} \mathfrak{a}\left(\sum_{(\Gamma', O') \in GO_{n,m}^{(1)}} w'_{(\Gamma', O')}^F B_{(\Gamma', O')}(\alpha)\right) &= \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{m!} \sum_{(\Delta, O_\Delta) \in GO_n^{(0)}} w_{(\Delta, O_\Delta)}^F(\varepsilon) \sum_{GO_{n,m}^{(1)} \ni (\Gamma, O) \supset (\Delta, O_\Delta)} \frac{l!}{k!} \varepsilon(s_O) \varepsilon(\tau_O) B_{(\Gamma, O)}(\alpha) \\ &= \frac{1}{m!} \sum_{(\Delta, O_\Delta) \in GO_n^{(0)}} w_{(\Delta, O_\Delta)}^F \sum_{GO_{n,m}^{(1)} \ni (\Gamma, O) \supset (\Delta, O_\Delta)} \frac{\ell!}{k!} \varepsilon(s_O) \varepsilon(\tau_O) B_{(\Gamma, O)}(\alpha). \end{aligned}$$

(ii) If F has type 2: then, since our graphs (Γ', O') have exactly one leg for each foot, F is isomorphic to $C_{n_1, m_1}^+ \times C_{n_2, m_2}^+$ with $n_2 > 0$ and $n_1 > 0$. This case corresponds to the subcase 1 of [AMM]. Let us suppose that $p_{i_1}, \dots, p_{i_{n_1}}$ and $q_{\ell+1}, \dots, q_{\ell+m}$ are collapsing on $q \in \mathbb{R}$. We note p_j the first aerial vertex of Γ' which is not a p_{i_s} and we impose $p_j = i (= \sqrt{-1})$. The other parameters are then fixed and we get a

parametrization of our configuration space $C_{n,m}^+$ by variables a_r, b_s, q_t (see the notation of [AMM]). We put $a_{i_1} = q$, $b = \Im(p_{i_1})$, and

$$p'_{i_k} = \frac{p_{i_k} - q}{b} \quad (2 \leq k \leq n_1), \quad q'_{\ell+r} = \frac{q_{\ell+r} - q}{b} \quad (1 \leq r \leq m_1).$$

That is $p_{i_k} = bp'_{i_k} + qb$ and $q_{\ell+r} = q'_{\ell+r} + qb$, and when b tends to zero, the p_{i_k} and the $q_{\ell+r}$ tend to q . We finally put

$$C_{n,m}^+(\varepsilon) = \{(p, q) \in C_{n,m}^+ : b = \varepsilon\}.$$

We get:

$$w'_{(\Gamma', O')}^F = \lim_{\varepsilon \rightarrow 0} \frac{k!}{|k|!} \int_{C_{n,m}^+(\varepsilon)} \omega_{(\Gamma', O')} := \lim_{\varepsilon \rightarrow 0} w'_{(\Gamma', O')}^F(\varepsilon).$$

Note that if Γ' has a bad edge, the weight $w'_{(\Gamma', O')}^F$ vanishes. We can thus consider also these graphs in our sum. Now, a similar computation as in (i) gives the result.

From now on, for any aerial oriented graph (Δ, O_Δ) in $GO_n^{(0)}$, note $C_{(\Delta, O_\Delta)}$ the operator $C_{(\Delta, O_\Delta)} : T_{poly}^{\otimes n}(\mathbb{R}^d) \rightarrow D_{poly}(\mathbb{R}^d)^{(1)} \simeq T_{poly}(\mathbb{R}^d)$ defined by

$$C_{(\Delta, O_\Delta)}(\alpha_1 \otimes \cdots \otimes \alpha_n) = \sum_{m \geq 0} \frac{1}{m!} \sum_{GO_{n,m}^{(1)} \ni (\Gamma, O) \supset (\Delta, O_\Delta)} \frac{\ell!}{k!} \varepsilon(s_O) \varepsilon(\tau_O) B_{(\Gamma, O)}(\alpha_1 \otimes \cdots \otimes \alpha_n)$$

where $\varepsilon(s_O), \varepsilon(\tau_O)$ have the same meaning as above.

Remark 3.1.

The definition of $C_{(\Delta, O_\Delta)}$ can be extended naturally to the space $GO_n'^{(0)}$ of aerial graphs (Δ', O'_Δ) with O'_Δ not necessarily compatible just by putting:

$$C_{(\Delta', O'_\Delta)} = \sum_{m \geq 0} \frac{1}{m!} \sum_{GO_{n,m}'^{(1)} \ni (\Gamma', O') \supset (\Delta', O'_\Delta)} \frac{\ell!}{k!} \varepsilon(s_{O'}) \varepsilon(\tau_{O'}) B_{(\Gamma', O')}.$$

We will need to use this extension in Section 5.

Suming up, we have

Proposition 3.4.

Consider the Kontsevich's explicit formality \mathcal{U} on \mathbb{R}^d . The formality equation can be read as follows:

$$F_n = E_n - d_H \mathcal{U}_n = 0,$$

and the skewsymmetrization of E_n has the form

$$\mathfrak{a} \circ E_n = \sum_{m \geq 0} \sum_{\substack{F \\ \text{face of} \\ \partial C_{n,m}^+}} \sum_{(\Gamma', O') \in GO_{n,m}'^{(1)}} w'_{(\Gamma', O')}^F B_{(\Gamma', O')},$$

where $w'_{(\Gamma', O')}^F = \int_{F \in \partial C_{n,m}^+} \omega'_{(\Gamma', O')}$. Then, for each face F ,

$$\mathfrak{a}\left(\sum_{(\Gamma', O') \in GO_{n,m}^{(1)}} w'_{(\Gamma', O')}^F B_{(\Gamma', O')}(\alpha)\right) = \sum_{(\Delta, O_\Delta) \in GO_n^{(0)}} w_{(\Delta, O_\Delta)}^F C_{(\Delta, O_\Delta)}(\alpha).$$

This proposition suggests to put

Definition 3.5.

A mapping φ from $T_{poly}(\mathbb{R}^d)^{\otimes n}$ to $D_{poly}(\mathbb{R}^d)^{(1)} \simeq T_{poly}(\mathbb{R}^d)$ will be called a K -graph mapping if it can be written

$$\varphi = \sum_{(\Delta, O_\Delta) \in GO_n^{(0)}} c_{(\Delta, O_\Delta)} C_{(\Delta, O_\Delta)}$$

with real coefficients $c_{(\Delta, O_\Delta)}$.

Such a mapping is homogeneous of degree s if $c_{(\Delta, O_\Delta)} = 0$ for all Δ such that

$$\# E(\Delta) + s \neq 2n - 2.$$

Definition 3.6.

A K -graph formality \mathcal{F} at order n is a graph formality up to order $n - 1$ such that

$$\varphi_n = \mathcal{F}_1^{-1} \circ \mathfrak{a} \circ E_n$$

is a K -graph mapping.

4. SYMMETRIZATION

4.1. Expressions for ∂ .

If B is a n -linear mapping $B : T_{poly}(\mathbb{R}^d)^{\otimes n} \rightarrow T_{poly}(\mathbb{R}^d)$, we define SB by

$$SB(\alpha_1 \otimes \cdots \otimes \alpha_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon_\alpha(\sigma) B(\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(n)}),$$

and say that B is symmetric if $SB = B$. Any symmetric mapping can be viewed as a map $\varphi : S^n(T_{poly}(\mathbb{R}^d)) \rightarrow T_{poly}(\mathbb{R}^d)$. With this symmetrization operator S , the expression of the Chevalley coboundary operator can be conveniently simplified. Indeed, we have

Proposition 4.1.

Let $\varphi : S^n(T_{poly}(\mathbb{R}^d)[1]) \rightarrow T_{poly}(\mathbb{R}^d)[1]$ be an n -cochain for ∂ , homogeneous with degree $|\varphi|$. Then, we can write

$$\partial\varphi = S(\tilde{\partial}\varphi)$$

where $\tilde{\partial}\varphi$ is given by :

$$\begin{aligned} \tilde{\partial}\varphi(\alpha_1 \otimes \cdots \otimes \alpha_{n+1}) &= (n+1)[\varphi(\alpha_1 \otimes \cdots \otimes \alpha_n) \bullet \alpha_{n+1} + \\ &\quad + (-1)^{|\varphi||\alpha_1|} \alpha_1 \bullet \varphi(\alpha_2 \otimes \cdots \otimes \alpha_{n+1}) + (-1)^{|\varphi|+1} n \varphi(\alpha_1 \bullet \alpha_2 \otimes \alpha_3 \otimes \cdots \otimes \alpha_{n+1})], \end{aligned}$$

or else by an expression imitating the Hochschild coboundary operator :

$$\begin{aligned} \tilde{\partial}\varphi(\alpha_1 \otimes \cdots \otimes \alpha_{n+1}) &= (n+1)[\varphi(\alpha_1 \otimes \cdots \otimes \alpha_n) \bullet \alpha_{n+1} + \\ &\quad + (-1)^{|\varphi|+1} \sum_{k=2}^{n+1} (-1)^{\sum_{s=1}^{k-2} |\alpha_s|} \varphi(\alpha_1 \otimes \cdots \otimes \alpha_{k-1} \bullet \alpha_k \otimes \cdots \otimes \alpha_{n+1}) \\ &\quad + (-1)^{|\varphi||\alpha_1|} \alpha_1 \bullet \varphi(\alpha_2 \otimes \cdots \otimes \alpha_{n+1})], \end{aligned}$$

Proof: By definition of ∂ , we have

$$\begin{aligned} \partial\varphi(\alpha_1 \dots \alpha_{n+1}) &= \sum_{i=1}^{n+1} \varepsilon_\alpha(1 \dots \hat{i} \dots n+1, i) \varphi(\alpha_1 \dots \hat{\alpha}_i \dots \alpha_{n+1}) \bullet \alpha_i + \\ &\quad + \sum_{i=1}^{n+1} (-1)^{|\varphi||\alpha_i|} \varepsilon_\alpha(i, 1 \dots \hat{i} \dots n+1) \alpha_i \bullet \varphi(\alpha_1 \dots \hat{\alpha}_i \dots \alpha_{n+1}) + \\ &\quad + \sum_{i \neq j} (-1)^{|\varphi|+1} \varepsilon_\alpha(ij, 1 \dots \widehat{ij} \dots n+1) \varphi(\alpha_i \bullet \alpha_j \alpha_1 \dots \widehat{\alpha_i \alpha_j} \dots \alpha_{n+1}) = \\ &= (1) + (2) + (3). \end{aligned}$$

Now, let us put

$$\begin{aligned} \psi_1(\alpha_1 \otimes \cdots \otimes \alpha_{n+1}) &= (n+1) \varphi(\alpha_1 \otimes \cdots \otimes \alpha_n) \bullet \alpha_{n+1} \\ \psi_2(\alpha_1 \otimes \cdots \otimes \alpha_{n+1}) &= (-1)^{|\varphi||\alpha_1|} (n+1) \alpha_1 \bullet \varphi(\alpha_2 \otimes \cdots \otimes \alpha_{n+1}) \\ \psi_3(\alpha_1 \otimes \cdots \otimes \alpha_{n+1}) &= (-1)^{|\varphi|+1} (n+1) n \varphi(\alpha_1 \bullet \alpha_2 \otimes \cdots \otimes \alpha_{n+1}) \\ \psi'_3(\alpha_1 \otimes \cdots \otimes \alpha_{n+1}) &= (-1)^{|\varphi|+1} \sum_{k=2}^{n+1} (-1)^{\sum_{s=1}^{k-2} |\alpha_s|} \varphi(\alpha_1 \otimes \cdots \otimes \alpha_{k-1} \bullet \alpha_k \otimes \cdots \otimes \alpha_{n+1}). \end{aligned}$$

First

$$\begin{aligned} S\psi_1(\alpha_1 \dots \alpha_{n+1}) &= \frac{(n+1)}{(n+1)!} \sum_{\sigma \in S_{n+1}} \varepsilon_\alpha(\sigma) \varphi(\alpha_{\sigma(1)} \dots \alpha_{\sigma(n)}) \bullet \alpha_{\sigma(n+1)} \\ &= \frac{(n+1)}{(n+1)!} \sum_{i=1}^{n+1} \sum_{\sigma: \sigma(n+1)=i} \varepsilon_\alpha(\sigma) \varphi(\alpha_{\sigma(1)} \dots \alpha_{\sigma(n)}) \bullet \alpha_i \\ &= \frac{(n+1)}{(n+1)!} \sum_{i=1}^{n+1} \sum_{\tau: \tau(i)=i} \varepsilon_\alpha(\tau \circ \sigma_i) \varphi(\alpha_{\tau(\sigma_i(1))} \dots \alpha_{\tau(\sigma_i(n))}) \bullet \alpha_i. \end{aligned}$$

Here σ_i is the permutation of S_{n+1} sending $(1, \dots, n+1)$ to $(1, \dots, \hat{i}, \dots, n+1, i)$. And, if we note $\bar{\tau}$ the restriction of τ to $\{1, \dots, \hat{i}, \dots, n+1\}$, we easily get:

$$\begin{aligned} S\psi_1(\alpha_1 \dots \alpha_{n+1}) &= \frac{n+1}{(n+1)!} \sum_{i=1}^{n+1} \sum_{\bar{\tau} \in S_n} \varepsilon_{\alpha \setminus \{\alpha_i\}}(\bar{\tau}) \varepsilon_{\alpha}(\sigma_i) \varphi(\alpha_{\bar{\tau}(1)} \dots \alpha_{\bar{\tau}(n+1)}) \bullet \alpha_i \\ &= \frac{(n+1)}{(n+1)!} n! \sum_{i=1}^{n+1} \varepsilon_{\alpha}(1 \dots \hat{i} \dots n+1, i) \varphi(\alpha_1 \dots \hat{\alpha}_i \dots \alpha_{n+1}) \bullet \alpha_i \\ &= (1). \end{aligned}$$

With exactly the same argument, we obtain:

$$S\psi_2(\alpha_1 \dots \alpha_{n+1}) = (2).$$

Now,

$$\begin{aligned} S\psi_3(\alpha_1 \dots \alpha_{n+1}) &= \\ &= \sum_{\sigma \in S_{n+1}} \frac{1}{(n+1)!} (-1)^{|\varphi|+1} \varepsilon_{\alpha}(\sigma) (n+1) n \varphi(\alpha_{\sigma(1)} \bullet \alpha_{\sigma(2)} \otimes \alpha_{\sigma(3)} \otimes \dots \otimes \alpha_{\sigma(n+1)}) \\ &= \sum_{i \neq j} \sum_{\substack{\sigma: \sigma(1)=i \\ \sigma(2)=j}} \varepsilon_{\alpha}(\sigma) \frac{1}{(n+1)!} (-1)^{|\varphi|+1} (n+1) n \varphi(\alpha_i \bullet \alpha_j \otimes \alpha_1 \otimes \alpha_{\sigma(3)} \otimes \dots \otimes \alpha_{\sigma(n+1)}) \\ &= \sum_{i \neq j} \sum_{\substack{\tau: \tau(i)=i \\ \tau(j)=j}} \varepsilon_{\alpha}(\tau) \frac{1}{(n+1)!} \varepsilon_{\alpha}(\sigma_{ij}) (-1)^{|\varphi|+1} (n+1) n \varphi(\alpha_i \bullet \alpha_j \otimes \alpha_{\tau(\sigma_{ij}(3))} \dots \\ &\quad \dots \otimes \alpha_{\tau(\sigma_{ij}(n+1))}), \end{aligned}$$

where σ_{ij} is the permutation of S_{n+1} sending $(1, \dots, n+1)$ to $(ij, 1 \dots \hat{i} \hat{j} \dots n+1)$. Now, if $\bar{\tau}$ denotes the restriction of τ to $\{1, \dots, \hat{i} \hat{j} \dots, n+1\}$, we get:

$$\begin{aligned} S\psi_3(\alpha_1 \dots \alpha_{n+1}) &= \\ &= \sum_{i \neq j} \frac{(-1)^{|\varphi|+1}}{(n-1)!} \sum_{\bar{\tau}: \tau(i)=i, \tau(j)=j} \varepsilon_{\alpha}(\sigma_{ij}) \varepsilon_{\alpha}(\bar{\tau}) \varphi(\alpha_i \bullet \alpha_j \otimes \alpha_{\bar{\tau}(1)} \otimes \dots \widehat{\alpha_i \alpha_j} \dots \otimes \alpha_{\bar{\tau}(n+1)}) \\ &= \sum_{i \neq j} (-1)^{|\varphi|+1} \varepsilon_{\alpha}(ij, 1 \dots \hat{i} \hat{j} \dots n+1) \varphi(\alpha_i \bullet \alpha_j \dots \widehat{\alpha_i \alpha_j} \dots \alpha_{n+1}) = (3). \end{aligned}$$

Finally,

$$\begin{aligned}
S\psi'_3(\alpha_1 \dots \alpha_{n+1}) &= \\
&= \frac{(n+1)}{(n+1)!} \sum_{\sigma \in S_{n+1}} (-1)^{|\varphi|+1} \sum_{k=2}^{n+1} (-1)^{\sum_{s=1}^{k-2} |\alpha_s|} \varepsilon_\alpha(\sigma) \varphi(\alpha_{\sigma(1)} \otimes \dots \\
&\quad \dots \otimes \alpha_{\sigma(k-2)} \otimes \alpha_{\sigma(k-1)} \bullet \alpha_{\sigma(k)} \otimes \dots \otimes \alpha_{\sigma(n+1)}) \\
&= \frac{(-1)^{|\varphi|+1}}{n!} \sum_{k=2}^{n+1} \sum_{\substack{i \neq j \\ \sigma: \sigma(k-1)=i, \\ \sigma(k)=j}} \sum_{\sigma} (-1)^{\sum_{s=1}^{k-2} |\alpha_s|} \varepsilon_\alpha(\sigma) \varphi(\alpha_{\sigma(1)} \dots \otimes \alpha_i \bullet \alpha_j \otimes \dots \otimes \alpha_{\sigma(n+1)}).
\end{aligned}$$

Let σ_{ij}^k be the permutation

$$\sigma_{ij}^k : (1 \dots n+1) \mapsto (1, \dots, k-2, i, j, k-1, k, \dots, n+1).$$

Then,

$$\begin{aligned}
(-1)^{|\varphi|+1} S\psi'_3(\alpha_1 \dots \alpha_{n+1}) &= \\
&= \frac{1}{n!} \sum_{\substack{2 \leq k \leq n+1 \\ i \neq j}} (-1)^{\sum_{s=1}^{k-2} |\alpha_s|} \varepsilon_\alpha(\sigma_{ij}^k) (n-1)! \varphi(\alpha_1 \otimes \dots \otimes \alpha_{(k-2)} \otimes \alpha_i \bullet \alpha_j \otimes \dots \otimes \alpha_{n+1}) \\
&= \frac{1}{n} \sum_{\substack{2 \leq k \leq n+1 \\ i \neq j}} (-1)^{\sum_{s=1}^{k-2} |\alpha_s|} \varepsilon_\alpha(\sigma_{ij}) \varepsilon_\alpha(\rho_{ij}^k) \varphi(\alpha_i \bullet \alpha_j \otimes \alpha_1 \otimes \dots \otimes \alpha_{n+1}) (-1)^{a_{ijk}}
\end{aligned}$$

with $a_{ijk} = (|\alpha_i| + |\alpha_j| + 1)(\sum_{s=1}^{k-2} |\alpha_s|)$. Here $\sigma_{ij} = (ij1 \dots \widehat{i} \dots n+1)$ and ρ_{ij}^k is the permutation

$$\rho_{ij}^k : (ij1 \dots \widehat{i} \dots n+1) \mapsto (1, \dots, k-2, i, j, k-1, k, \dots, n+1),$$

we have thus used the composition $\sigma_{ij}^k = \rho_{ij}^k \circ \sigma_{ij}$. Now, since

$$\varepsilon_\alpha(\rho_{ij}^k) = (-1)^{(|\alpha_i| + |\alpha_j|)(\sum_{s=1}^{k-2} |\alpha_s|)},$$

we get

$$S\psi'_3(\alpha_1 \dots \alpha_{n+1}) = \sum_{i \neq j} (-1)^{|\varphi|+1} \varepsilon_\alpha(\sigma_{ij}) \varphi(\alpha_i \bullet \alpha_j \otimes \alpha_1 \otimes \dots \otimes \alpha_{n+1}) = (3).$$

This ends the proof.

4.2. Symmetrization on graphs.

We want now to describe the symmetrization directly on the space of graphs. Since we are mainly interested in K -graph formalities, we will restrict ourselves to linear combinations of graphs for which the associated operator is a K -graph mapping (see Subsection 3.3).

4.2.1. S_n action on $GO_{n,m}$ and $GO_n^{(0)}$.

There is a natural action of S_n on $GO_{n,m}$ and $GO_n^{(0)}$ that we shall now define. Let σ be a permutation in S_n . Let (Γ, O) be in $GO_{n,m}$, for the moment, note P_i the set $star(p_i)$, ordered by O . Let σ_Γ be the permutation of the ordered set $E(\Gamma)$ of edges of Γ sending $P_1 \cup \dots \cup P_n$ to $P_{\sigma(1)} \cup \dots \cup P_{\sigma(n)}$. We denote by $\varepsilon_\Gamma(\sigma_\Gamma)$ the sign of σ_Γ and by $\sigma(\Gamma, O) := (\sigma(\Gamma), \sigma(O))$ the graph with aerial vertices $p'_1 = p_{\sigma(1)}, \dots, p'_n = p_{\sigma(n)}$ oriented by $\sigma_\Gamma(E(\Gamma))$ (see Figure 2). We apply the same definition for aerial graphs in $GO_n^{(0)}$. Clearly, σ sends $GO_{n,m}$ (and $GO_n^{(0)}$) onto itself.

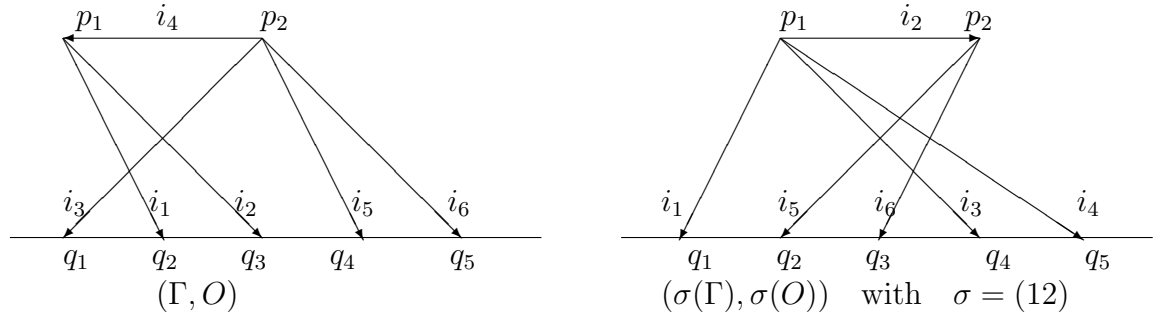


Figure 2

This S_n action on $GO_{n,m}^{(1)}$ is entirely different from the action of S_m defined in Section 3. But there is an analog of Lemma 3.1:

Lemma 4.2. *For all σ in S_n , for any (Γ, O) in $GO_{n,m}^{(1)}$ and for all polyvector fields α_i ,*

$$B_{(\sigma(\Gamma), \sigma(O))}(\alpha_1 \otimes \dots \otimes \alpha_n) = B_{(\Gamma, O)}(\alpha_{\sigma(1)} \otimes \dots \otimes \alpha_{\sigma(n)}).$$

Proof: With our notations

$$B_{(\Gamma, O)}(\alpha_{\sigma(1)} \otimes \dots \otimes \alpha_{\sigma(n)})(f_1, \dots, f_m) = \sum_{1 \leq i_{t_1} \dots i_{t_{|k|}} \leq d} \prod_{i=1}^n \partial_{\text{end}(p_i)} \alpha_{\sigma(i)}^{P_i} \prod_{j=1}^m \partial_{\text{end}(q_j)} f_j.$$

Since the permutation σ_Γ does not affect the order inside each P_i , we have

$$\begin{aligned}
B_{(\Gamma, O)}(\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(n)})(f_1, \dots, f_m) &= \\
&= \sum_{1 \leq i_{t_1} \dots i_{t_{|k|}} \leq d} \prod_{i=1}^n \partial_{\text{end}(p_{\sigma(i)})} \alpha_{\sigma(i)}^{P_{\sigma(i)}} \prod_{j=1}^m \partial_{\text{end}(q_j)} f_j \\
&= \sum_{1 \leq i_{t_1} \dots i_{t_{|k|}} \leq d} \prod_{i'=1}^n \partial_{\text{end}(p'_{i'})} \alpha_{i'}^{P_{i'}} \prod_{j=1}^m \partial_{\text{end}(q_j)} f_j \\
&= B_{\sigma(\Gamma, O)}(\alpha_1 \otimes \cdots \otimes \alpha_n)(f_1, \dots, f_m).
\end{aligned}$$

4.2.2. Symmetrization for K -graph mappings.

Definition 4.3.

Let $(\delta, O_\delta) = \sum_{(\Delta, O_\Delta) \in GO_n^{(0)}} c_{(\Delta, O_\Delta)}(\Delta, O_\Delta)$ be a linear combination of aerial graphs with n vertices. We say that (δ, O_δ) is symmetric if

$$c_{(\sigma(\Delta), \sigma(O_\Delta))} = \varepsilon_\Delta(\sigma_\Delta) c_{(\Delta, O_\Delta)} \quad \forall (\Delta, O_\Delta), \forall \sigma \in S_n.$$

Proposition 4.4.

If $(\delta, O_\delta) = \sum_{(\Delta, O_\Delta) \in GO_n^{(0)}} c_{(\Delta, O_\Delta)}(\Delta, O_\Delta)$ is symmetric, then the corresponding K -graph mapping

$$C_{(\delta, O_\delta)} = \sum_{(\Delta, O_\Delta) \in GO_n^{(0)}} c_{(\Delta, O_\Delta)} C_{(\Delta, O_\Delta)}$$

is symmetric.

Proof: Let σ be in S_n and let $\alpha_1, \dots, \alpha_n$ be n polyvector fields on \mathbb{R}^d . Then, by Lemma 4.1 and using the fact that δ is symmetric,

$$\begin{aligned}
C_{(\delta, O_\delta)}(\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(n)}) &= \\
&= \sum_{(\Delta, O_\Delta)} c_{(\Delta, O_\Delta)} \sum_{m \geq 0} \frac{1}{m!} \sum_{(\Gamma, O) \supset (\Delta, O_\Delta)} \frac{\ell!}{k!} \varepsilon(s_O) \varepsilon(\tau_O) B_{(\sigma(\Gamma), \sigma(O))}(\alpha_1 \otimes \cdots \otimes \alpha_n) \\
&= \sum_{\sigma^{-1}(\Delta, O_\Delta)} c_{(\sigma^{-1}(\Delta), \sigma^{-1}(O_\Delta))} \sum_{m \geq 0} \frac{1}{m!} \sum_{\sigma^{-1}(\Gamma, O) \supset \sigma^{-1}(\Delta, O_\Delta)} \\
&\quad \frac{\ell!}{k!} \varepsilon(s_{\sigma^{-1}(O)}) \varepsilon(\tau_{\sigma^{-1}(O)}) B_{(\Gamma, O)}(\alpha_1 \otimes \cdots \otimes \alpha_n)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{(\Delta, O_\Delta)} \varepsilon_\Delta(\sigma_\Delta) c_{(\Delta, O_\Delta)} \sum_{m \geq 0} \frac{1}{m!} \sum_{(\Gamma, O) \supset (\Delta, O_\Delta)} \\
&\quad \frac{\ell!}{k!} \varepsilon(s_{\sigma^{-1}(O)}) \varepsilon(\tau_{\sigma^{-1}(O)}) B_{(\Gamma, O)}(\alpha_1 \otimes \cdots \otimes \alpha_n).
\end{aligned}$$

Extending σ_Δ to $E(\Gamma)$ in the obvious way, we can write

$$\tau_O \circ s_O = \sigma_\Delta \circ \tau_{\sigma^{-1}(O)} \circ s_{\sigma^{-1}(O)} \circ \sigma_\Gamma^{-1}.$$

Thus,

$$\begin{aligned}
&C_\delta(\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(n)}) = \\
&= \sum_{(\Delta, O_\Delta)} c_{(\Delta, O_\Delta)} \sum_{m \geq 0} \frac{1}{m!} \sum_{(\Gamma, O) \supset (\Delta, O_\Delta)} \varepsilon_\Gamma(\sigma_\Gamma) \frac{l!}{k!} \varepsilon(s_O) \varepsilon(\tau_O) B_{(\Gamma, O)}(\alpha_1 \otimes \cdots \otimes \alpha_n).
\end{aligned}$$

Since each $\varepsilon_\Gamma(\sigma_\Gamma)$ clearly coincides with the sign $\varepsilon_\alpha(\sigma)$ of σ , we get

$$C_\delta(\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(n)}) = \varepsilon_\alpha(\sigma) C_\delta(\alpha_1 \otimes \cdots \otimes \alpha_n).$$

This proves the result.

5. CHEVALLEY COHOMOLOGY FOR GRAPHS

We want now to prove that, on K -graph mappings, the Chevalley coboundary operator can be nicely reduced to an operator acting on purely aerial graphs.

5.1. Purely aerial and compatible oriented graphs.

For any (Δ, O_Δ) in $GO_n^{(0)}$ with vertices $p_1 < \cdots < p_n$, we still note $\ell_i = \# \text{star}^\Delta(p_i)$. We also put $|\Delta| = \sum \ell_i = |\ell|$.

Let us now fix two indexes $i \neq j$. We shall say that an aerial graph $(\Delta', O_{\Delta'})$ in $GO_{n+1}'^{(0)}$ ($O_{\Delta'}$ not necessarily compatible) with vertices $p'_1 < \cdots < p'_{n+1}$ reduces to (Δ, O_Δ) in the indexes i, j if the two following assertions hold:

- (i) the vertices p'_i and p'_j of Δ' are linked by only the edge $\vec{p'_i p'_j}$
- (ii) the new graph $(\Delta'_{ij}, O_{\Delta'_{ij}})$, obtained by gluing together the vertices p'_i, p'_j of Δ' , by suppressing the edge $\vec{p'_i p'_j}$ and considering the induced ordering, coincides with (O, Δ) .

Moreover, we shall say that $(\Delta', O_{\Delta'})$ reduces properly to (Δ, O_Δ) in the indexes i, j , if $(\Delta', O_{\Delta'})$ reduces to (Δ, O_Δ) in the indexes i, j and if

$$\inf(\# \text{star}^{\Delta'}(p'_i) + \# \text{end}^{\Delta'}(p'_i), \# \text{star}^{\Delta'}(p'_j) + \# \text{end}^{\Delta'}(p'_j)) > 1.$$

Note this by $(\Delta', O_{\Delta'}) \rightarrow_{i,j} (\Delta, O_\Delta)$ (resp. $(\Delta', O_{\Delta'}) \xrightarrow{prop}_{i,j} (\Delta, O_\Delta)$). We will use the same notation for graphs (Γ, O) in $GO_{n,m}^{(1)}$.

Let us introduce

Definition 5.1.

If (Δ, O_Δ) is an aerial oriented graph in $GO_n^{(0)}$, we define the coboundary $\partial(\Delta, O_\Delta)$ of (Δ, O_Δ) by:

$$\partial(\Delta, O_\Delta) = (-1)^{|\Delta|+1} \sum_{i \neq j} \sum_{(\Delta', O_{\Delta'}) \rightarrow_{i,j}^{prop} (\Delta, O_\Delta)} \varepsilon(\Delta', O_{\Delta'}, \Delta, O_\Delta)(\Delta', O_{\Delta'}).$$

Here, $\varepsilon(\Delta', O_{\Delta'}, \Delta, O_\Delta)$ is the sign of the permutation of $E(\Delta')$, which consists in putting first the edge $\vec{p'_i p'_j}$, then the other edges starting from p'_i (with the ordering induced by $O_{\Delta'}$), then the edges starting from p'_j (also with the induced ordering), and finally all the remaining edges (with the ordering given by O_Δ).

We extend ∂ by linearity to all combinations $(\delta, O_\delta) = \sum_{(\Delta, O_\Delta)} c_{(\Delta, O_\Delta)}(\Delta, O_\Delta)$. Note that the restriction of ∂ to symmetric combinations of graphs is an operator of cohomology.

More precisely, we can prove:

Proposition 5.2.

With the same notations as above and for any symmetric combination of graphs (δ, O_δ) , we have

$$\partial(C_{(\delta, O_\delta)}) = C_{\partial(\delta, O_\delta)}.$$

Proof: First, we remark that $C_{(\Delta, O_\Delta)}$ is a linear combination of m -differential operators $B_{(\Gamma, O)}(\alpha)$ with for k_i -vector fields α_i :

$$m - 2 = |B_{(\Gamma, O)}(\alpha_1 \otimes \cdots \otimes \alpha_n)| = \sum_{i=1}^n |\alpha_i| + |B_{(\Gamma, O)}| = \sum_{i=1}^n k_i - 2n + |B_{(\Gamma, O)}|,$$

here $||$ stands for the degree in $T_{poly}(\mathbb{R}^d)[1]$ and $D_{poly}(\mathbb{R}^d)[1]$. Now, since the graphs (Γ, O) occurring in $C_{(\Delta, O_\Delta)}$ are 1-graphs, $k_i = \ell_i + m_i$, for each i and $m = \sum_{i=1}^n m_i$. Thus,

$$|B_{(\Gamma, O)}| = \sum_{i=1}^n \ell_i = |\Delta| \mod 2.$$

Now, by definition of ∂ on operators:

$$\begin{aligned} \partial C_{(\Delta, O_\Delta)}(\alpha_1 \dots \alpha_{n+1}) &= \sum_{j=1}^{n+1} \varepsilon_\alpha(1 \dots \hat{j} \dots n+1, j) C_{(\Delta, O_\Delta)}(\alpha_1 \dots \hat{\alpha}_j \dots \alpha_{n+1}) \bullet \alpha_j \\ &+ \sum_{i=1}^{n+1} (-1)^{|\Delta||\alpha_i|} \varepsilon_\alpha(i, 1 \dots \hat{i} \dots n+1) \alpha_i \bullet C_{(\Delta, O_\Delta)}(\alpha_1 \dots \hat{\alpha}_i \dots \alpha_{n+1}) \\ &+ \sum_{i \neq j} (-1)^{|\Delta|+1} \varepsilon_\alpha(ij, 1 \dots \widehat{ij} \dots n+1) C_{(\Delta, O_\Delta)}(\alpha_i \bullet \alpha_j \alpha_1 \dots \widehat{\alpha_i \alpha_j} \dots \alpha_{n+1}) \\ &= (i) + (ii) + (iii). \end{aligned}$$

Let us first consider the term (iii). We have:

$$\begin{aligned} C_{(\Delta, O_\Delta)}(\alpha_i \bullet \alpha_j \dots \widehat{\alpha_i \alpha_j} \dots \alpha_{n+1}) &= \\ &= \sum_{m \geq 0} \frac{1}{m!} \sum_{GO_{n,m}^{(1)} \ni (\Gamma, O) \supset (\Delta, O_\Delta)} \frac{\ell!}{k!} \varepsilon(s_O) \varepsilon(\tau_O) B_{(\Gamma, O)}(\alpha_i \bullet \alpha_j \dots \widehat{\alpha_i \alpha_j} \dots \alpha_{n+1}). \end{aligned}$$

Now, we can write (see [AMM] for details)

$$B_{(\Gamma, O)}(\alpha_i \bullet \alpha_j \alpha_1 \dots \alpha_{n+1}) = \sum_{(\Gamma', O') \rightarrow_{i,j} (\Gamma, O)} (-1)^{\ell_{\Gamma'}-1} B_{(\Gamma', O')}(\alpha_1 \dots \alpha_{n+1}),$$

here $\ell_{\Gamma'}$ denotes the position of the edge $\vec{p'_i p'_j}$ in Γ' , and the sign $(-1)^{\ell_{\Gamma'}-1}$ comes directly from the definition of \bullet .

Now, let us consider a graph (Γ', O') which reduces to (Γ, O) in the indexes i, j . We permute the edges as follows: we put at the first position the edge $\vec{p'_i p'_j}$, then the other edges starting from p'_i , then the edges starting from p'_j , and finally we put all the legs at the end in the order of the feet. This gives a sign which can be written as

$$\varepsilon_\alpha(ij, 1 \dots \widehat{ij} \dots n+1) (-1)^{\ell_{\Gamma'}-1} \varepsilon(s_O) \varepsilon(\tau_O).$$

Starting from (Γ', O') , one can also put first the legs at the end in the order of the feet, then put at the beginning the aerial edges starting from p'_i and those starting from p'_j , and finally put the aerial edge $\vec{p'_i p'_j}$ at the first position. If we denote by Δ' the aerial part of Γ' and by $\ell_{\Delta'}$ the position of the edge $\vec{p'_i p'_j}$ in Δ' , the resulting sign is

$$\varepsilon(s_{O'}) \varepsilon(\tau_{O'}) \varepsilon(\Delta', \Delta) (-1)^{\ell_{\Delta'}-1}.$$

These two permutations of the edges of Γ' obviously coincide, thus

$$\varepsilon_\alpha(ij, 1 \dots \widehat{ij} \dots n+1) (-1)^{\ell_{\Gamma'}-1} \varepsilon(s_O) \varepsilon(\tau_O) = \varepsilon(s_{O'}) \varepsilon(\tau_{O'}) \varepsilon(\Delta', \Delta) (-1)^{\ell_{\Delta'}-1}.$$

It follows that:

$$\begin{aligned}
& C_{(\Delta, O_\Delta)}(\alpha_i \bullet \alpha_j \dots \widehat{\alpha_i \alpha_j} \dots \alpha_{n+1}) = \\
& = \sum_{m \geq 0} \frac{1}{m!} \sum_{GO_{n,m}^{(1)} \ni (\Gamma, O) \supset (\Delta, O_\Delta)} \varepsilon_\alpha(ij1 \dots n+1) \left(\sum_{(\Gamma', O') \rightarrow_{i,j} (\Gamma, O)} \frac{\ell!}{k!} \right. \\
& \quad \left. \varepsilon(s_{O'}) \varepsilon(\tau_{O'}) (-1)^{\ell_{\Delta'}-1} \varepsilon(\Delta', \Delta) B_{(\Gamma', O')}(\alpha_1 \dots \alpha_{n+1}) \right) = \\
& = \varepsilon_\alpha(ij, 1 \dots n+1) \sum_{m \geq 0} \frac{1}{m!} \left(\sum_{(\Delta', O_{\Delta'}) \rightarrow_{i,j} (\Delta, O_\Delta)} (-1)^{\ell_{\Delta'}-1} \varepsilon(\Delta', \Delta) \sum_{(\Gamma', O') \supset (\Delta', O_{\Delta'})} \frac{\ell!}{k!} \right. \\
& \quad \left. \varepsilon(s'_O) \varepsilon(\tau'_O) B_{(\Gamma', O')}(\alpha_1 \dots \alpha_{n+1}) \right) = \\
& = \varepsilon_\alpha(ij, 1 \dots n+1) \sum_{(\Delta', O_{\Delta'}) \rightarrow_{i,j} (\Delta, O_\Delta)} (-1)^{\ell_{\Delta'}-1} \varepsilon(\Delta', \Delta) C_{(\Delta', O_{\Delta'})}(\alpha_1 \dots \alpha_{n+1}).
\end{aligned}$$

Finally,

$$\begin{aligned}
(iii) & = (-1)^{|\Delta|+1} \sum_{(\Delta', O_{\Delta'}) \rightarrow_{i,j} (\Delta, O_\Delta)} (-1)^{\ell_{\Delta'}-1} \varepsilon(\Delta', \Delta) C_{(\Delta', O_{\Delta'})}(\alpha_1 \dots \alpha_{n+1}) \\
& = (-1)^{|\Delta|+1} \sum_{(\Delta', O_{\Delta'}) \rightarrow_{i,j} (\Delta, O_\Delta)} \varepsilon(\Delta', O_{\Delta'}, \Delta, O_\Delta) C_{(\Delta', O_{\Delta'})}(\alpha_1 \dots \alpha_{n+1}).
\end{aligned}$$

Now, let $(\delta, O_\delta) = \sum c_{(\Delta, O_\Delta)}(\Delta, O_\Delta)$ be a symmetric combination of graphs and let us put:

$$C_{(\delta, O_\delta)} = (i)_\delta + (ii)_\delta + (iii)_\delta.$$

We have to prove that $-((i)_\delta + (ii)_\delta)$ coincides with the non-proper terms of $(iii)_\delta$ that is with:

$$\sum_{(\Delta, O_\Delta)} c_{(\Delta, O_\Delta)} (-1)^{|\Delta|+1} \sum_{i \neq j} \sum_{(\Delta', O_{\Delta'}) \rightarrow_{i,j}^{non-prop} (\Delta, O_\Delta)} (-1)^{\ell_{\Delta'}-1} \varepsilon(\Delta', O_{\Delta'}) (\Delta', O_{\Delta'}).$$

Let us consider first the term $(ii)_\delta$:

$$(ii)_\delta = \sum_{(\Delta, O_\Delta)} c_{(\Delta, O_\Delta)} \sum_{i=1^n} (-1)^{|\Delta|+|\alpha_i|} \varepsilon_\alpha(i, 1 \dots n+1) \alpha_i \bullet C_{(\Delta, O_\Delta)}(\alpha_1 \dots \hat{\alpha}_i \dots \alpha_{n+1}).$$

We identify $C_{(\Delta, O_\Delta)}(\alpha)$ with a polyvector field, and put:

$$C_{(\Delta, O_\Delta)}(\alpha_1 \dots \hat{\alpha}_i \dots \alpha_{n+1}) = [C_{(\Delta, O_\Delta)}(\alpha_1 \dots \hat{\alpha}_i \dots \alpha_{n+1})]^{r_1 \dots r_m} \partial_{r_1} \wedge \dots \wedge \partial_{r_m}.$$

Thus:

$$\begin{aligned}
& \alpha_i \bullet C_{(\Delta, O_\Delta)}(\alpha_1 \dots \hat{\alpha}_i \dots \alpha_{n+1}) = \\
& = \sum_{j \neq i} \sum_{l \leq k_i} (-1)^{l-1} \alpha_i^{i_1 \dots s \dots k_i-1} [C_{(\Delta, O_\Delta)}(\alpha_1 \dots \partial_s(\alpha_j) \dots \hat{\alpha}_i \dots \alpha_{n+1})]^{r_1 \dots r_m} \\
& \quad \partial_{i_1} \wedge \dots \wedge \partial_{i_{k_i-1}} \wedge \partial_{r_1} \wedge \dots \wedge \partial_{r_m}.
\end{aligned}$$

Let σ be the permutation $(j1 \dots \widehat{ij} \dots n+1)$ and $(\Delta^\sigma, O_{\Delta^\sigma})$ be the aerial graph obtained by relabelling the vertices of Δ in the ordering given by σ . Then

$$C_{(\Delta, O_\Delta)}(\alpha_1 \dots \partial_s(\alpha_j) \dots \widehat{\alpha_i} \dots \alpha_{n+1}) = C_{(\Delta^\sigma, O_{\Delta^\sigma})}(\partial_s(\alpha_j) \alpha_1 \dots \widehat{\alpha_i \alpha_j} \dots \alpha_{n+1}).$$

But (δ, O_δ) is symmetric, thus:

$$c_{(\Delta^\sigma, O_{\Delta^\sigma})} = c_{(\Delta, O_\Delta)} \varepsilon_\alpha(j, 1.. \widehat{ij} \dots n+1).$$

Hence,

$$(ii)_\delta = \sum_{(\Delta, O_\Delta)} \sum_{i \neq j} (-1)^{|\Delta| |\alpha_i|} \varepsilon_\alpha(ij1 \dots n+1) c_{(\Delta, O_\Delta)} \sum_{\ell \leq k_i} (-1)^{\ell-1} \alpha_i^{i_1 \dots s \dots i_{k_i-1}} \\ [C_{(\Delta, O_\Delta)}(\partial_s \alpha_j \alpha_1 \dots \widehat{\alpha_i \alpha_j} \dots \alpha_{n+1})]^{r_1 \dots r_m} \partial_{i_1} \wedge \dots \wedge \partial_{i_{k_i-1}} \wedge \partial_{r_1} \wedge \dots \wedge \partial_{r_m}.$$

It is now easy to see that $-(ii)_\delta$ coincides with non-proper terms of $(iii)_\delta$ more precisely with the non-proper terms corresponding to the graphs Δ' with

$$(\Delta', O_{\Delta'}) \rightarrow_{i,j} (\Delta, O_\Delta) \quad \text{and} \quad (\# \text{ star}^{\Delta'}(p'_i) + \# \text{ end}^{\Delta'}(p'_i)) = 1.$$

(In this case, $\ell_{\Delta'} = 1$.) In the same way, one can check that $-(i)_\delta$ coincides with the remaining non-proper terms of $(iii)_\delta$ that is with the non-proper terms corresponding to the case:

$$(\Delta', O_{\Delta'}) \rightarrow_{i,j} (\Delta, O_\Delta) \quad \text{and} \quad (\# \text{ star}^{\Delta'}(p'_j) + \# \text{ end}^{\Delta'}(p'_j)) = 1.$$

The result follows.

5.2. Purely aerial and non-oriented graphs.

We shall say that a graph is non-oriented if there is an ordering only on the aerial vertices but no ordering on the edges of the graph. We are now interested in translating our cohomology on non-oriented graphs. Let Δ be an aerial non-oriented graph with n vertices $p_1 < \dots < p_n$. We still note $\ell_i = \text{star}^\Delta(p_i)$ and $\ell! = \ell_1! \dots \ell_n!$. We put the lexicographical ordering that is

$$\vec{ab} \leq \vec{a'b'} \quad \text{if and only if} \quad (a = a' \text{ and } a < b') \text{ or } (a < a')$$

on the edges of Δ . This yields a compatible ordering on Δ , called the standard ordering. We note $(\Delta, O_\Delta^{\text{std}})$ the resulting oriented graph.

Now, let us put

$$\Delta = \frac{1}{\ell!} \sum_{O_\Delta: (\Delta, O_\Delta) \in GO_n^{(0)}} \varepsilon(\sigma_{(O_\Delta^{\text{std}}, O_\Delta)})(\Delta, O_\Delta).$$

By definition of ∂ on compatible oriented graphs, we have:

$$\begin{aligned} \partial\Delta = \frac{1}{\ell!} & \left(\sum_{O_\Delta: (\Delta, O_\Delta) \in GO_n^{(0)}} \varepsilon(\sigma_{(O_\Delta^{std}, O_\Delta)}) (-1)^{|\Delta|+1} \right. \\ & \left. \sum_{i \neq j} \sum_{(\Delta', O_{\Delta'}) \rightarrow_{i,j}^{prop} (\Delta, O_\Delta)} \varepsilon(\Delta', O_{\Delta'}, \Delta, O_\Delta) (\Delta', O_{\Delta'}) \right). \end{aligned}$$

Note that the sign

$$\tilde{\varepsilon}(\Delta, \Delta') := \varepsilon(O_\Delta^{std}, O_\Delta) \varepsilon(\Delta', O_{\Delta'}, \Delta, O_\Delta) \varepsilon(O_{\Delta'}^{std}, O_{\Delta'})$$

does not depend on O_Δ and $O_{\Delta'}$. This yields a very simple expression for the coboundary $\partial\Delta$ of Δ :

$$\partial\Delta = \frac{1}{\ell!} \sum_{i \neq j} \sum_{\Delta' \supset \Delta} \tilde{\varepsilon}(\Delta', \Delta) \Delta'.$$

We extend ∂ to linear combination of graphs $\delta = \sum_{\Delta} c_{\Delta} \Delta$.

Now, if Δ is a non-oriented graph with vertices $p_1 < \dots < p_n$ and if σ is a permutation in S_n , we denote by $\sigma(\Delta)$ the non-oriented graph with vertices $p_{\sigma(1)} < \dots < p_{\sigma(n)}$. A linear combination $\delta = \sum_{\Delta} c_{\Delta} \Delta$ of non-oriented graphs with n labelled vertices is said to be symmetric if for any σ in S_n , we have $c_{\Delta} = c_{\sigma(\Delta)}$. Our operator ∂ restricted to symmetric δ is clearly an operator of cohomology.

More precisely, let us put, for an aerial non-oriented graph Δ ,

$$C_{\Delta} = \frac{1}{\ell!} \sum_{O_\Delta: (\Delta, O_\Delta) \in GO_n^{(0)}} \varepsilon(\sigma_{(O_\Delta^{std}, O_\Delta)}) C_{(\Delta, O_\Delta)}.$$

We extend this definition by linearity to all linear combinations. Then, by computations similar to those we did before for oriented graphs, we can prove:

Proposition 5.3. *For any symmetric combination $\delta = \sum_{\Delta} c_{\Delta} \Delta$ of graphs with n labelled vertices, we have*

$$\partial(C_{\delta}) = C_{\partial(\delta)}.$$

5.3. Examples.

Let Δ_1 be the graph with only one vertex p_1 . Let α_1 be a k_1 -vector field. Then,

$$C_{\Delta_1}(\alpha_1) = \frac{1}{(k_1!)^2} \sum_{GO_{n,m}^{(1)} \ni (\Gamma, O) \supset \Delta_1} \varepsilon(s_O) \varepsilon(\tau_O) B_{(\Gamma, O)}(\alpha_1).$$

There is only one graph occurring in this sum, namely the graph Γ with one aerial vertex p_1 , k_1 terrestrial vertices q_1, \dots, q_{k_1} and k_1 edges: $p_1 \vec{q}_1, \dots, p_1 \vec{q}_{k_1}$. For any

σ in S_{k_1} , we denote by (Γ, O^σ) the graph Γ endowed with the ordering given by $p_1 \vec{q}_{\sigma(1)} \dots p_1 \vec{q}_{\sigma(k_1)}$. Clearly,

$$C_{\Delta_1}(\alpha_1) = \frac{1}{(k_1!)^2} \sum_{\sigma \in S_{k_1}} \varepsilon(\sigma) B_{(\Gamma, O^\sigma)}(\alpha_1) = \mathcal{F}_1^{(0)}(\alpha_1) \simeq \alpha_1,$$

and C_{Δ_1} just corresponds to the identity mapping Id .

Now, let Δ_2 be the aerial graph with two vertices $p_1 < p_2$ and one edge $p_1 \vec{p}_2$. Let α_1 be a k_1 -vector field and α_2 a k_2 -vector field. Then,

$$C_{\Delta_2}(\alpha_1 \otimes \alpha_2) = \frac{1}{(k_1 + k_2 - 1)!} \sum_{(\Gamma, O) \supset \Delta_2, (\Gamma, O) \in GO_{n,m}^{(1)}} \frac{1}{k_1! k_2!} \varepsilon(s_O) \varepsilon(\tau_O) B_{(\Gamma, O)}(\alpha_1 \otimes \alpha_2).$$

There are exactly $\frac{(k_1+k_2-1)!}{(k_1-1)!k_2!}$ graphs Γ containing Δ_2 and having exactly $(k_1 - 1)$ legs starting from p_1 and k_2 legs starting from p_2 . For each of them, we choose a compatible ordering. There are $k_1!k_2!$ possibilities to do it. Thus, there are exactly $k_1(k_1 + k_2 - 1)!$ compatible oriented graphs (Γ, O) occurring in C_{Δ_2} . For each of these graphs, $\varepsilon(s_O)$ corresponds to the permutation of S_{k_1} which consists in putting the aerial edge of (Γ, O) at the first position and $\varepsilon(\tau_O)$ corresponds to the permutation of $S_{k_1+k_2-1}$ which consists in putting the legs in the order of the feet. There is thus $k_1(k_1 + k_2 - 1)!$ terms in C_{Δ_2} ; each of these terms looks like:

$$\frac{1}{(k_1 + k_2 - 1)! k_1! k_2!} \varepsilon(s_O) \varepsilon(\tau_O) B_{(\Gamma, O)} = \frac{1}{(k_1 + k_2 - 1)! k_1! k_2!} (-1)^{\ell-1} \varepsilon(\sigma) \alpha_1^{i_{\sigma(1)} \dots i_{\sigma(\ell-1)} s i_{\sigma(\ell)} \dots i_{\sigma(k_1-1)}} \partial_s (\alpha_2^{i_{\sigma(k_1)} \dots i_{\sigma(k_1+k_2-1)}}) \partial_{i_{\sigma(1)}} \otimes \dots \otimes \partial_{i_{s(k_1+k_2-1)}}.$$

Thus, we have

$$C_{\Delta_2}(\alpha_1 \otimes \alpha_2) = \mathcal{F}_1^{(0)}(\alpha_1 \bullet \alpha_2) \simeq \alpha_1 \bullet \alpha_2.$$

Now, let us consider the aerial graph Δ_2^- with two vertices $p_1 < p_2$ and one edge: $p_2 \vec{p}_1$. In the same way as above, one can see that

$$C_{\Delta_2^-}(\alpha_1 \otimes \alpha_2) = (-1)^{k_1 k_2} \alpha_2 \bullet \alpha_1.$$

In other words, $C_{\Delta_2 + \Delta_2^-}$ coincides with Q_2 .

The identity map Id and Q_2 are thus easy examples of K -graph mappings and the fact that Q_2 is the Chevalley coboundary of Id can be directly checked on the graphs. Indeed, we have with our notations:

$$\partial \Delta_1 = \tilde{\varepsilon}(\Delta_2, \Delta_1) \Delta_2 + \tilde{\varepsilon}(\Delta_2^-, \Delta_1) \Delta_2^- = \Delta_2 + \Delta_2^-.$$

Hence,

$$Q_2 = C_{\Delta_2 + \Delta_2^-} = C_{\partial \Delta_1} = \partial C_{\Delta_1} = \partial Id.$$

6. TRIVIALITY OF THE COHOMOLOGY FOR SMALL n

Our first example proves that the first cohomology group H^1 is trivial, since, for $n = 1$, there is only one purely aerial graph, namely Δ_1 .

Suppose now $n = 2$, there is one graph Δ with two vertices and with degree 0 $|\Delta| = 0$, the non connected symmetric graph denoted $\Delta_1 \times \Delta_1$ without any edges. Its coboundary does not vanish, with evident notations, we have :

$$\partial(\Delta_1 \times \Delta_1) = S[(\Delta_2^+ + \Delta_2^-) \times \Delta_1 + \Delta_1 \times (\Delta_2^+ + \Delta_2^-)] \neq 0.$$

In degree 1 ($|\Delta| = 1$), there is only one symmetrized graph, $\Delta_2^+ + \Delta_2^-$. Our second example shows this graph is a coboundary.

Finally there is no graph with degree larger than 1, indeed, the number of edges for a graph with 2 vertices is at most 2, but there is only one graph Δ with $|\Delta| = 2$, the graph $\Delta_{2,2}$



But the symmetrization of this graph is $\Delta_{2,2} - \Delta_{2,2} = 0$. Thus the second cohomology group H^2 vanishes

It is possible to prove with elementary arguments that $H^3 = 0$ too. For that, we consider the different cases, $|\Delta| = 0, \dots, 6$, then we define the order of a graph in the following way:

We define the order o_i of a vertex p_i as the pair (ℓ_i, r_i) of number ℓ_i of edges starting from p_i and the number r_i of edges ending at p_i , we shall say that $o = (\ell, r)$ is smaller than $o' = (\ell', r')$ and note $o < o'$ if and only if $\ell + r < \ell' + r'$ or $\ell + r = \ell' + r'$ and $\ell < \ell'$.

We define then the order $o(\Delta)$ of a graph Δ as $o(\Delta) = (o_1, \dots, o_n)$ if Δ has n vertices. The order $o(\delta)$ of a linear combination $\delta = \sum c_\Delta \Delta$ of graphs is the maximum of $o(\Delta)$ for $c_\Delta \neq 0$ for the lexicographic ordering. We define the symbol of δ as:

$$symb(\delta) = \sum_{o(\Delta)=o(\delta)} c_\Delta C_\Delta.$$

Case 1 $|\Delta| = 0$

There is only one graph, disconnected and symmetric, the graph $\Delta_1 \times \Delta_1 \times \Delta_1$, it is not a cocycle since

$$\partial(\Delta_1 \times \Delta_1 \times \Delta_1) = S((\Delta_2^+ + \Delta_2^-) \times \Delta_1 \times \Delta_1) \neq 0.$$

Case 2 $|\Delta| = 1$

There is, up to the ordering of vertices, only one symmetrized, disconnected graph: $\delta = S(\Delta_2^+ \times \Delta_1)$. This graph is a coboundary:

$$\partial(\Delta_1 \times \Delta_1) = \frac{1}{3}S((\Delta_2^+ + \Delta_2^-) \times \Delta_1) = \frac{2}{3}\delta.$$

Case 3 $|\Delta| = 2$

There is, up to the ordering of vertices, a disconnected graph $\Delta_{2,2} \times \Delta_1$ and three connected graphs, the following graphs (we choose the ordering of vertices which maximizes the order, then for a given order, maximizes, for the lexicographic ordering, the set $E(\Delta)$ of edges of graphs Δ):

$$\begin{aligned}\Delta_{3,2,1}; E(\Delta_{3,2,1}) &= \{p_1\vec{p}_2, p_1\vec{p}_3\} \\ \Delta_{3,2,2}; E(\Delta_{3,2,2}) &= \{p_2\vec{p}_1, p_1\vec{p}_3\} \\ \Delta_{3,2,3}; E(\Delta_{3,2,3}) &= \{p_2\vec{p}_1, p_3\vec{p}_1\}.\end{aligned}$$

After symmetrization, we get $S(\Delta_{2,2} \times \Delta_1) = 0$, $S(\Delta_{3,2,1}) = S(\Delta_{3,2,3}) = 0$ and

$$\text{symb}(S(\Delta_{3,2,2})) = \frac{1}{6}\Delta_{3,2,2}, \quad o(S(\Delta_{3,2,2})) = ((1, 1), (1, 0), (0, 1)).$$

When we compute $\partial(S(D))$, we have to consider the blow-up of each vertex of each graph in $S(\Delta)$. If the vertex p has order $o = (\ell, r)$, then we get a few graphs with two vertices p' and p'' at the place of p , these vertices have order $o' = (\ell', r')$, $o'' = (\ell'', r'')$ with conditions:

$$\ell' + r' \geq 2, \quad \ell'' + r'' \geq 2, \quad \ell' + \ell'' = \ell + 1, \quad r' + r'' = r + 1.$$

Then we look for $o(\partial\Delta)$. If $r > 0$, then the maximal possible order among those (o', o'') is $((\ell + 1, r - 1), (0, 2))$, if $r = 0$, it is $((\ell, r), (1, 1)) = ((\ell, 0), (1, 1))$.

Thus $o(\partial(S(D_{3,2,2}))) \leq ((2, 0), (0, 2), (1, 0), (0, 1))$ but more precisely:

$$\text{symb}(\partial(S(\Delta_{3,2,2}))) = \frac{1}{6}\Delta', \quad E(\Delta') = \{p_1\vec{p}_2, p_1\vec{p}_4, p_3\vec{p}_2\}$$

and, since there is only one graph in the symbol,

$$o(\partial(S(\Delta_{3,2,2}))) = ((2, 0), (0, 2), (1, 0), (0, 1)).$$

No vector in this case is a cocycle, ∂ is an one-to-one mapping.

Case 4 $|\Delta| = 3$

From now on, all our graphs are connected. We repeat the argument of preceeding case, we get the following results:

They are, up to a permutation of vertices, four graphs:

$$\begin{aligned}\Delta_{3,3,1}; E(\Delta_{3,3,1}) &= \{p_1\vec{p}_2, p_1\vec{p}_3, p_2\vec{p}_1\} \\ \Delta_{3,3,2}; E(\Delta_{3,3,2}) &= \{p_1\vec{p}_2, p_2\vec{p}_1, p_3\vec{p}_1\} \\ \Delta_{3,3,3}; E(\Delta_{3,3,3}) &= \{p_1\vec{p}_2, p_1\vec{p}_3, p_2\vec{p}_3\} \\ \Delta_{3,3,4}; E(\Delta_{3,3,4}) &= \{p_1\vec{p}_2, p_2\vec{p}_3, p_3\vec{p}_1\}\end{aligned}$$

Their symmetrization do not vanish, we get:

$$\begin{aligned}o(S(\Delta_{3,3,1})) &= ((2, 1), (1, 1), (0, 1)), o(\partial(S(\Delta_{3,3,1}))) = ((3, 0), (1, 1), (0, 2), (0, 1)) \\ o(S(\Delta_{3,3,2})) &= ((1, 2), (1, 1), (1, 0)), o(\partial(S(\Delta_{3,3,2}))) = ((2, 1), (1, 1), (0, 2), (0, 1)) \\ o(S(\Delta_{3,3,3})) &= ((2, 0), (1, 1), (0, 2)), o(\partial(S(\Delta_{3,3,3}))) = ((2, 0), (2, 0), (0, 2), (0, 2)) \\ o(S(\Delta_{3,3,4})) &= ((1, 1), (1, 1), (1, 1)), o(\partial(S(\Delta_{3,3,4}))) = ((2, 0), (1, 1), (1, 1), (0, 2))\end{aligned}$$

Then ∂ is still a one-to-one mapping on that space of graphs.

Case 5 $|\Delta| = 4$

They are, up to a permutation of vertices, four graphs:

$$\begin{aligned}\Delta_{3,4,1}; E(\Delta_{3,4,1}) &= \{p_1\vec{p}_2, p_1\vec{p}_3, p_2\vec{p}_1, p_3\vec{p}_1\} \\ \Delta_{3,4,2}; E(\Delta_{3,4,2}) &= \{p_1\vec{p}_2, p_1\vec{p}_3, p_2\vec{p}_1, p_2\vec{p}_3\} \\ \Delta_{3,4,3}; E(\Delta_{3,4,3}) &= \{p_1\vec{p}_2, p_1\vec{p}_3, p_2\vec{p}_1, p_3\vec{p}_2\} \\ \Delta_{3,4,4}; E(\Delta_{3,4,4}) &= \{p_1\vec{p}_2, p_2\vec{p}_1, p_3\vec{p}_1, p_3\vec{p}_2\}\end{aligned}$$

Their symmetrization do not vanish, we get:

$$\begin{aligned}o(S(\Delta_{3,4,1})) &= ((2, 2), (1, 1), (1, 1)), o(\partial(S(\Delta_{3,4,1}))) = ((3, 1), (1, 1), (1, 1), (0, 2)) \\ o(S(\Delta_{3,4,2})) &= ((2, 1), (2, 1), (0, 2)), o(\partial(S(\Delta_{3,4,2}))) = ((3, 0), (2, 1), (0, 2), (0, 2)) \\ o(S(\Delta_{3,4,3})) &= ((2, 1), (1, 2), (1, 1)), o(\partial(S(\Delta_{3,4,3}))) = ((3, 0), (1, 2), (1, 1), (0, 2)) \\ o(S(\Delta_{3,4,4})) &= ((1, 2), (1, 2), (2, 0)), o(\partial(S(\Delta_{3,4,4}))) = ((2, 1), (1, 2), (2, 0), (0, 2))\end{aligned}$$

Then ∂ is still a one-to-one mapping on that space of graphs.

Case 6 $|\Delta| = 5$

Up to a permutation of vertices, this space contains only one graph :

$$\Delta_{3,5,1}; E(\Delta_{3,5,1}) = \{p_1\vec{p}_2, p_1\vec{p}_3, p_2\vec{p}_1, p_2\vec{p}_3, p_3\vec{p}_1\}.$$

Its symmetrization does not vanish and:

$$o(S(\Delta_{3,5,1})) = ((2, 2), (2, 1), (1, 2)), o(\partial(S(\Delta_{3,5,1}))) = ((3, 1), (2, 1), (1, 2), (0, 2)).$$

Then ∂ is still a one-to-one mapping on that space of graphs.

Case 7 $|\Delta| = 6$

In the last case, there is only one graph:

$$\Delta_{3,6,1}; \quad E(\Delta_{3,6,1}) = \{p_1\vec{p}_2, p_1\vec{p}_3, p_2\vec{p}_1, p_2\vec{p}_3, p_3\vec{p}_1, p_3\vec{p}_2\}.$$

But its symmetrization does vanish.

This prove:

Proposition 6.1.

The three first spaces H^1 , H^2 and H^3 of the Chevalley cohomology for graphs vanish.

7. CANONICAL COCYCLES FOR THE LINEAR CASE

Let us first recall the construction of the relevant cocycles for the cohomology of the Lie algebra of vector fields $\mathcal{X}(\mathbb{R}^d)$ associated to the Lie derivative of smooth functions, see for instance [DWL] for an explicit presentation of this cohomology.

A basis of the Lie algebra $\bigwedge^{inv}(\mathfrak{gl}(d, \mathbb{R}))$ of multilinear, totally antisymmetric, invariant forms on $\mathfrak{gl}(d, \mathbb{R})$ is given by:

$$\zeta^{(j_1)} \wedge \dots \wedge \zeta^{(j_q)} \quad j_k \text{ odd}, \quad j_1 < j_2 < \dots < j_q < 2d$$

where the mapping $\zeta^{(j)}$ are the mapping:

$$\zeta^{(j)}(A_1, \dots, A_j) = \mathfrak{a}(Tr(A_1 \dots A_j)).$$

Then, for each odd n , the linear form θ defined on $\bigwedge^n \mathcal{X}(\mathbb{R}^d)$ by:

$$\theta(\xi_1, \xi_2, \dots, \xi_n) = \zeta^{(n)}(Jac(\xi_1), \dots, Jac(\xi_n))$$

is a cocycle for the coboundary operator associated to the Lie derivative:

$$\begin{aligned} d\theta(\xi_0 \dots \xi_n) &= \sum_{i=0}^n (-1)^i \mathcal{L}_{\xi_i} \theta(\xi_0 \dots \hat{\xi}_i \dots \xi_n) + \\ &+ \frac{1}{2} \sum_{i \neq j} (-1)^{i+j} \theta([\xi_i, \xi_j] \cdot \xi_0 \dots \widehat{\xi_i \xi_j} \dots \xi_n). \end{aligned}$$

This cocycle is not a coboundary (see [DWL]).

Let Ψ be a n -cochain on $T_{poly}(\mathbb{R}^d)$ with value in the space $T_{poly}(\mathbb{R}^d)^{-1}$ (i.e. in $C^\infty(\mathbb{R}^d)$), ψ its restriction to $\mathcal{X}(\mathbb{R}^d)$. Then the restriction of $\partial\Psi$ to $\mathcal{X}(\mathbb{R}^d)$ is exactly $d\psi$.

For instance, we consider the wheel without axis-graph Δ :

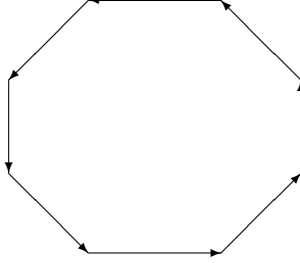


Figure 3

Denote δ its symmetrization, it defines a cochain $\Psi = C_\delta$, by construction, on vector fields ξ_i , we get

$$\begin{aligned} \psi(\xi_1 \dots \xi_n) &= \Psi(\xi_1 \dots \xi_n) = C_d(\xi_1 \dots \xi_n) \\ &= \frac{1}{n!} \sum_{(\sigma \in S_n)} \varepsilon(\sigma) \partial_{i_n} \xi_{\sigma(1)}^{i_1} \partial_{i_1} \xi_{\sigma(2)}^{i_2} \dots \partial_{i_{n-1}} \xi_{\sigma(n)}^{i_n} \\ &= \theta(\xi_1 \dots \xi_n). \end{aligned}$$

Thus

$$\begin{aligned} C_{\partial\delta}(\xi_0 \dots \xi_n) &= \partial C_\delta(\xi_0 \dots \xi_n) = \partial \Psi(\xi_0 \dots \xi_n) \\ &= d\theta(\xi_0 \dots \xi_n) = 0. \end{aligned}$$

Let us restrict ourselves to the space of linear polyvector-fields. This is a subalgebra of $T_{poly}(\mathbb{R}^d)$ equipped with the Schouten bracket, thus we can restrict our coboundary operator to cochains defined on this subalgebra. We get a new operator ∂_{lin} . Our previous computation tells us that the graph happening in $\partial\delta$ are on the following forms:

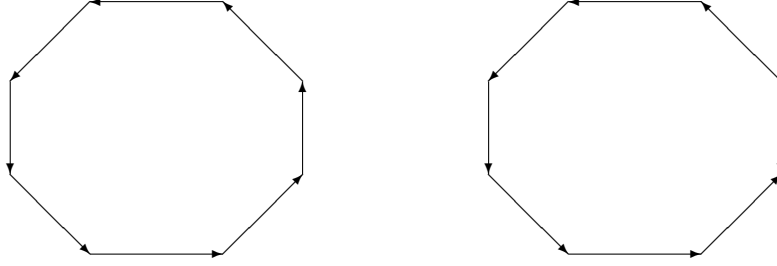


Figure 4

For linear polyvector fields, only the first case appears. Then $B_{\partial_{lin}(\delta)}(\alpha_0 \dots \alpha_n)$ vanishes if one of the α_j is not a vector fields. And

$$B_{\partial_{lin}(\delta)}(\xi_0 \dots \xi_n) = C_{\partial\delta}(\xi_0 \dots \xi_n) = 0.$$

Since the mapping $\gamma \mapsto B_\gamma$ is one to one, $\partial_{lin}\delta = 0$.

Now, if δ would be a coboundary $d = \partial_{lin}\beta$, then β has $n - 1$ vertices, $n - 1$ edges. To each vertex ends exactly one edge. If there is a vertex p from which no edge is starting, note $\vec{p}p$ the edge ending at p . Since the graphs in β can be deduced from the graphs $\partial_{lin}\beta$ only by proper reduction, there is no reduction at the vertex p and in $\partial_{lin}\beta$, there remains a unique edge $\vec{p}p$. But there is no such graph in δ , thus we can eliminate in β all the graphs with a vertex without edge starting (we consider only graphs 'without hand'). Now from each vertex of a graph in β , there is exactly one edge starting. As previously, the restriction of $\partial\beta$ to the vector fields coincides with $\partial_{lin}\beta$ and

$$\begin{aligned} dC_\beta(\xi_0 \dots \xi_n) &= \partial C_\beta(\xi_0 \dots \xi_n) = C_{\partial\beta}(\xi_0 \dots \xi_n) \\ &= C_{\partial_{lin}\beta}(\xi_0 \dots \xi_n) = C_\delta(\xi_0 \dots \xi_n) \\ &= \theta(\xi_0 \dots \xi_n). \end{aligned}$$

This is impossible.

Thus each of the wheel without axis with an odd number of vertices Δ gives rise to a canonical true cocycle for ∂_{lin} .

Remark 7.1. Suppose we want to build a linear formality \mathcal{F} from the space of linear polyvector fields to the space of multi differential operators. As we saw in Section

2, the obstruction for such a construction is a mapping φ , of degree 1, with $n \geq 4$ arguments. Such a mapping corresponds to purely aerial graphs with n vertices and $2n - 3$ edges, in the linear case, we should have $2n - 3 \leq n$, this is impossible. Every linear formality at order n can be extended to a linear formality.

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